PERIYAR UNIVERSITY

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SALEM - 636 011, Tamil Nadu, India.

CENTRE FOR DISTANCE AND ONLINE EDUCATION
(CDOE)

MASTER OF SCIENCE IN MATHEMATICS
SEMESTER - II

ELECTIVE COURSE: DIFFERENCE EQUATIONS
(Candidates admitted from 2024 onwards)

PERIYAR UNIVERSITY

CENTRE FOR DISTANCE AND ONLINE EDUCATION (CDOE)

M.Sc. MATHEMATICS 2024 admission onwards

ELECTIVE – VII
Difference Equations

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Contents

SYLLABUS: DIFFERENCE EQUATIONS

Objectives:

Difference equations usually describe the evolution of certain phenomena over the course of time. The aim of studying this course is

- to introduce the difference calculus
- to study linear difference equations and to know how to solve them
- to know the stability theory for homogeneous linear system of difference equations
- to study the asymptotic behavior of solutions of homogeneous linear difference equations

UNIT I: Difference Calculus Difference operator - Summation – Generating functions and approximate summation.

UNIT II: Linear Difference Equations First order equations - General results for linear equations - Solving linear equations.

UNIT III: Linear Difference Equations(continuation) Equations with variable coefficients – The z -transform.

UNIT IV: Stability Theory Initial value problems for linear systems – Stability of linear systems.

UNIT V: Asymptotic Methods Introduction – Asymptotic analysis of sums – Linear equations.

References:

1. W.G. Kelley and A.C. Peterson, “Difference Equations”, 2nd Edition, Academic Press, New York, 2001.

Suggested Readings:

1. R.P. Agarwal, “Difference Equations and Inequalities”, 2nd Edition, Marcel Dekker, New York, 2000.
2. S.N. Elaydi, “An Introduction to Difference Equations”, 3rd Edition, Springer, India, 2008.
3. R. E. Mickens, “Difference Equations”, 3rd Edition, CRC Press, 2015.

UNIT 1

Unit 1

The Difference Calculus

Objectives:

This unit briefly surveys the most important aspects of the difference calculus. It deals with the difference operator and the computation of sums, introduces the concept of generating function, and contains a proof of the important Euler summation formula.

1.1 The Difference Operator

Definition 1.1.1. *Let $y(t)$ be a function of a real or complex variable t. The "difference operator" Δ is defined by*

$$\Delta y(t) = y(t+1) - y(t).$$

We can take the domain of y to be a set of consecutive integers such as the natural numbers $N = \{1, 2, \cdots\}$.

Remark 1.1.2. *The step size of one unit used in the definition is not really a restriction. Consider a difference operation with a step size $h > 0$ say, $z(s+h) - z(s)$. If we take $y(t) = z(th)$, then we have*

$$\begin{aligned} z(s+h) - z(s) &= z(th+h) - z(th) \\ &= y(t+1) - y(t) \\ &= \Delta y(t). \end{aligned}$$

When applying the difference operator to a function of two or more variables, a subscript will be used to indicate which variable is to be shifted by one unit.

For example,

$$\Delta_t te^n = (t+1)e^n - te^n = e^n,$$

while

$$\Delta_n te^n = te^{n+1} - te^n = te^n(e-1).$$

Remark 1.1.3. *Higher order differences are defined by composing the difference operator with itself. The second order difference is*

$$\begin{aligned} \Delta^2 y(t) &= \Delta\big(\Delta y(t)\big) \\ &= \Delta\big(y(t+1) - y(t)\big) \\ &= \big(y(t+2) - y(t+1)\big) - 2q\big(y(t+1) - y(t)\big) \\ \Rightarrow \Delta^2 y(t) &= y(t+2) - 2y(t+1) + y(t). \end{aligned}$$

In general, the formula for the n^{th} order difference is given by

$$\begin{aligned} \Delta^n y(t) &= y(t+n) - ny(t+n-1) + \frac{n(n-1)}{2!} y(t+n-2) + \cdots + (-1)^n y(t) \quad (1.1) \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} y(t+n-k). \end{aligned}$$

Definition 1.1.4. *The "shift operator" E is defined by*

$$Ey(t) = y(t+1).$$

If I denotes the identity operator defined by

$$Iy(t) = y(t).$$

Then, we have

$$\Delta = E - I.$$

Theorem 1.1.5. *The following are some fundamental properties of Δ:*

(a) $\Delta^m(\Delta^n y(t)) = \Delta^{m+n} y(t)$, *for all positive integers m and n.*

(b) $\Delta(y(t) + z(t)) = \Delta y(t) + \Delta z(t)$.

(c) $\Delta(Cy(t)) = C\Delta y(t)$, *if C is a constant.*

(d) $\Delta(y(t)z(t)) = y(t)\Delta z(t) + Ez(t)\Delta y(t)$.

(e) $\Delta \frac{y(t)}{z(t)} = \frac{z(t)\Delta y(t) - y(t)\Delta z(t)}{z(t)Ez(t)}$.

Proof. (a) For all positive integers m and n, we have

$$\begin{aligned}\Delta^m(\Delta^n y(t)) &= (E-I)^m\big((E-I)^n y(t)\big)\\ &= (E-I)^{m+n}y(t)\\ &= \Delta^{m+n}y(t).\end{aligned}$$

(b) We have

$$\begin{aligned}\Delta(y(t)+z(t)) &= (y(t+1)+z(t+1))-(y(t)+z(t))\\ &= (y(t+1)-y(t))+(z(t+1)-z(t))\\ &= \Delta y(t)+\Delta z(t).\end{aligned}$$

(c) For a constant C, we have

$$\begin{aligned}\Delta(Cy(t)) &= Cy(t+1)-Cy(t)\\ &= C(y(t+1)-y(t))\\ &= C\Delta y(t).\end{aligned}$$

(d) We have

$$\begin{aligned}\Delta(y(t)z(t)) &= y(t+1)z(t+1)-y(t)z(t)\\ &= y(t+1)z(t+1)-y(t)z(t+1)+y(t)z(t+1)-y(t)z(t)\\ &= z(t+1)(y(t+1)-y(t))+y(t)(z(t+1)-z(t))\\ &= z(t+1)\Delta y(t)+y(t)\Delta z(t)\\ &= Ez(t)\Delta y(t)+Ey(t)\Delta z(t).\end{aligned}$$

(e) We have

$$\begin{aligned}\Delta\left(\frac{y(t)}{z(t)}\right) &= \frac{y(t+1)}{z(t+1)}-\frac{y(t)}{z(t)}\\ &= \frac{y(t+1)z(t)-y(t)z(t+1)}{z(t+1)z(t)}\\ &= \frac{y(t+1)z(t)-y(t)z(t+1)-y(t)z(t)+y(t)z(t)}{z(t)Ez(t)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{z(t)[y(t+1)-y(t)]-y(t)[z(t+1)-z(t)]}{z(t)Ez(t)} \\
&= \frac{z(t)\Delta y(t)-y(t)\Delta z(t)}{z(t)Ez(t)}.
\end{aligned}
$$

□

The following theorem give formulas for differences of some basic functions.

Theorem 1.1.6. *Let "a" be a constant. Then*

(a) $\Delta a^t = (a-1)a^t$.

(b) $\Delta \sin at = 2\sin\frac{a}{2}\cos a\left(t+\frac{1}{2}\right)$.

(c) $\Delta \cos at = -2\sin\frac{a}{2}\sin a\left(t+\frac{1}{2}\right)$.

(d) $\Delta \log at = \log\left(1+\frac{1}{t}\right)$.

(e) $\Delta \log\Gamma(t) = \log t$.

(Here $\log t$ *represents any logarithm of the positive number* t*.)*

Proof. (a) We have

$$\Delta a^t = a^{t+1}-a^t = (a-1)a^t.$$

(b) We have

$$
\begin{aligned}
\Delta \sin at &= \sin a(t+1) - \sin at \\
&= 2\sin\left(\frac{a(t+1)-at}{2}\right)\cos\left(\frac{a(t+1)+at}{2}\right) \\
&= 2\sin\frac{a}{2}\ \cos\frac{2at+a}{2} \\
&= 2\sin\frac{a}{2}\cos a\left(t+\frac{1}{2}\right).
\end{aligned}
$$

(c) We have

$$
\begin{aligned}
\Delta \cos at &= \cos a(t+1) - \cos at \\
&= -2\sin\left(\frac{a(t+1)-at}{2}\right)\sin\left(\frac{a(t+1)+at}{2}\right) \\
&= -2\sin\frac{a}{2}\ \sin a\left(t+\frac{1}{2}\right).
\end{aligned}
$$

(d) We have

$$\begin{aligned} \Delta \log at &= \log a(t+1) - \log at \\ &= \log(at+a) - \log at \\ &= \log\left(\frac{at+a}{at}\right) \\ &= \log\left(1+\frac{1}{t}\right). \end{aligned}$$

(e) We have

$$\begin{aligned} \Delta \log \Gamma(t) &= \log \Gamma(t+1) - \log \Gamma(t) \\ &= \log \frac{\Gamma(t+1)}{\Gamma(t)} \\ &= \log \frac{t\Gamma(t)}{\Gamma(t)} \\ &= \log t. \end{aligned}$$

□

Note: All the formulas in Theorem 1.1.6 remain valid if a constant shift is introduced in the 't' variable.

For example:

$$\Delta a^{t+k} = a^{t+k+1} - a^{t+k} = (a-1)a^{t+k}.$$

Example 1.1.7. *Compute* $\Delta \sec \pi t$.

First, let us compute $\Delta \sec \pi t$ using Theorem 1.1.5(e).

We know that $\Delta \frac{y(t)}{z(t)} = \frac{z(t)\Delta y(t) - y(t)\Delta z(t)}{z(t)Ez(t)}$.

Then

$$\begin{aligned} \Delta \sec \pi t &= \Delta \frac{1}{\cos \pi t} \\ &= \frac{(\cos \pi t)(\Delta 1) - (1)(\Delta \cos \pi t)}{\cos \pi t \cos \pi(t+1)} \\ &= \frac{2 \sin \frac{\pi}{2} \sin \pi(t+\frac{1}{2})}{\cos \pi t \cos \pi(t+1)} \end{aligned}$$

$$\begin{aligned} &= \frac{2(\sin \pi t \cos \frac{\pi}{2} + \cos \pi t \sin \frac{\pi}{2})}{\cos \pi t(\cos \pi t \cos \pi - \sin \pi t \sin \pi)} \\ &= \frac{2 \cos \pi t}{(\cos \pi t)(-\cos \pi t)} \\ &= -2 \sec \pi t. \end{aligned}$$

(b) Next, let us compute $\Delta \sec \pi t$ using the definition of Δ .

$$\begin{aligned} \Delta \sec \pi t &= \sec \pi(t+1) - \sec \pi t \\ &= \frac{1}{\cos \pi(t+1)} - \frac{1}{\cos \pi t} \\ &= \frac{1}{\cos \pi t \cos \pi - \sin \pi t \sin \pi} - \frac{1}{\cos \pi t} \\ &= \frac{1}{-\cos \pi t} - \frac{1}{\cos \pi t} \\ &= -2 \sec \pi t. \end{aligned}$$

Note:

$$\begin{aligned} \Delta_t t^n &= (t+1)^n - t^n \\ &= \sum_{k=0}^{n} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) t^k - t^n \\ &= \sum_{k=0}^{n-1} (-1)^k \left(\begin{array}{c} n \\ k \end{array} \right) t^k. \end{aligned}$$

Definition 1.1.8. *The "falling factorial power" $t^{\underline{r}}$ is defined as follows, according to the value of r.*

(a) If $r = 1, 2, 3, \cdots$ then $t^{\underline{r}} = t(t-1)(t-2)\cdots(t-r+1)$.

(b) If $r = 0$ then $t^{\underline{0}} = 1$.

(c) If $r = -1, -2, -3, \cdots$, then $t^{\underline{r}} = \frac{1}{(t+1)(t+2)\cdots(t-r)}$.

(d) If r is not an integer, then

$$t^{\underline{r}} = \frac{\Gamma(t+1)}{\Gamma(t-r+1)}.$$

Remark 1.1.9. *It is understood that the definition of $t^{\underline{r}}$ is given only for those values of t and r that makes the formulas meaningful. For example, $(-2)^{-3}$ is not defined since the expression in part (c) involves division by zero and $(\frac{1}{2})^{\frac{3}{2}}$ is meaningless because Γ(0) is undefined.*

Remark 1.1.10. *If r is a positive integer, then*

$$\begin{aligned}\frac{\Gamma(t+1)}{\Gamma(t-r+1)} &= \frac{t\Gamma(t)}{\Gamma(t-r+1)} \\ &= \frac{t(t-1)\Gamma(t-1)}{\Gamma(t-r+1)} \\ &= \cdots \frac{t(t-1)\cdots(t-r+1)\Gamma(t-r+1)}{\Gamma(t-r+1)} \\ &= t(t-1)\cdots(t-r+1).\end{aligned}$$

So (a) is a special case of (d). In a similar way, (b) and (c) are particular cases of (d).

Definition 1.1.11. *The "binomial coefficient "$\begin{pmatrix} t \\ r \end{pmatrix}$ is defined by*

$$\begin{pmatrix} t \\ r \end{pmatrix} = \frac{t^{\underline{r}}}{\Gamma(r+1)},$$

where t and r are positive integers with $t \geq r$.

Theorem 1.1.12. *(a) $\Delta_t t^{\underline{r}} = r t^{\underline{r-1}}$.*

(b) $\Delta_t \binom{t}{r} = \binom{t}{r-1}, \quad (r \neq 0)$.

(c) $\Delta_t \binom{r+t}{t} = \binom{r+t}{t+1}$.

Proof. (a) Before we consider the general cases, let's prove (a) for a positive integer r.

$$\begin{aligned}\Delta_t t^{\underline{r}} &= (t+1)^{\underline{r}} - t^{\underline{r}} \\ &= (t+1)(t)(t-1)\cdots(t-r+2) - t(t-1)(t-2)\cdots(t-r+1) \\ &= t(t-1)\cdots(t-r+2)[(t+1)+(t-r+1)] \\ &= r t^{\underline{r-1}}.\end{aligned}$$

Now, Let r be arbitrary. From (d) of Definition 1.1.8, we have

$$\begin{aligned}\Delta_t t^{\underline{r}} &= \Delta_t \frac{\Gamma(t+1)}{\Gamma(t-r+1)} = \frac{\Gamma(t+2)}{\Gamma(t-r+2)} - \frac{\Gamma(t+1)}{\Gamma(t-r+1)} \\ &= \frac{(t+1)\Gamma(t+1)}{\Gamma(t-r+2)} - \frac{(t-r+1)\Gamma(t+1)}{\Gamma(t-r+2)}\end{aligned}$$

$$
\begin{aligned}
&= \frac{\Gamma(t+1)}{\Gamma(t-r+2)}\left[(t+1)-(t-r+1)\right] \\
&= \frac{\Gamma(t+1)}{\Gamma(t-r+2)}(r) \\
&= rt^{\underline{r-1}}.
\end{aligned}
$$

(b) We know that

$$\binom{t}{r} = \frac{t^{\underline{r}}}{\Gamma(r+1)}.$$

Then

$$
\begin{aligned}
\Delta_t \binom{t}{r} &= \Delta_t\left(\frac{t^{\underline{r}}}{\Gamma(r+1)}\right) \\
&= \frac{rt^{\underline{r-1}}}{\Gamma(r+1)} \\
&= \frac{rt^{\underline{r-1}}}{r\Gamma(r)} \\
&= \frac{t^{\underline{r-1}}}{\Gamma(r)} \\
&= \binom{t}{r-1}.
\end{aligned}
$$

(c) Consider

$$
\begin{aligned}
\Delta_t \binom{r+t}{t} &= \binom{r+t+1}{t+1} - \binom{r+t}{t} \\
&= \Delta_t \binom{r+t+1-1}{t+1} + \binom{r+t+1-1}{t+1-1} - \binom{r+t}{t} \\
&= \binom{r+t}{t+1}.
\end{aligned}
$$

□

Example 1.1.13. *Find a solution to the difference equation*

$$y(t+2) - 2y(t+1) + y(t) = t(t-1).$$

The given difference equation can be written in the form,

$$\Delta^2 y(t) = t^{\underline{2}}.$$

From Theorem 1.1.12, we know that $\Delta_t t^{\underline{r}} = r t^{\underline{r-1}}$*. Then*

$$\begin{aligned}\Delta^2 t^{\underline{4}} &= \Delta\left(\Delta t^{\underline{4}}\right)\\ &= \Delta 4 t^{\underline{3}}\\ &= 12 t^{\underline{2}}.\end{aligned}$$

So $y(t) = \frac{t^{\underline{4}}}{12}$ *is a solution of the difference equation.*

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Difference operator and its properties
2. Relation between difference operator and shift operator
3. Difference of some elementary functions
4. Falling factorial power

Check your Progress:

1. The difference operator Δ is defined by $\Delta y(t) =$ ————

   (A) $y(t+1)$ (B) $y(t+1) - y(t)$ (C) $y(t+1) + y(t)$ (D) None of these

2. The value of $\Delta_t t e^n$ is

   (A) $(t+1)e^n$ (B) e^n (C) e^{n+1} (D) te^n

3. The value of Δa^t is

   (A) $(a-1)a^t$ (B) a^{t+1} (C) $(a+1)a^t$ (D) a^t

1.2 Summation

Definition 1.2.1. *An "indefinite sum" (or "antidifference") of* $y(t)$*, denoted by* $\sum y(t)$*, is any function so that*

$$\Delta\left(\sum y(t)\right) = y(t)$$

for all t *in the domain of* y*.*

Example 1.2.2. *Compute the indefinite sum* $\sum 6^t$.

We know that $\Delta a^t = a^t(a-1)$. *Then*

$$\Delta 6^t = 5 \cdot 6^t.$$

$$\Rightarrow \Delta\big(\frac{6^t}{5}\big) = 6^t.$$

It follows that $\frac{6^t}{5}$ *is an indefinite sum of* 6^t.

Let us find all other indefinite sums of 6^t.

Let $C(t)$ *be a function with the same domain as* 6^t *so that* $\Delta C(t) = 0$. *Then*

$$\Delta\left(\frac{6^t}{5} + C(t)\right) = \Delta\left(\frac{6^t}{5}\right) + \Delta C(t) = 6^t,$$

so $\frac{6^t}{5} + C(t)$ *is an indefinite sum of* 6^t.

Further, if $f(t)$ *is any indefinite sum of* 6^t, *then*

$$\Delta\left(f(t) - \frac{6^t}{5}\right) = \Delta f(t) - \Delta\big(\frac{6^t}{5}\big) = 6^t - 6^t = 0,$$

so $f(t) = \frac{6^t}{5} + C(t)$ *for some* $C(t)$ *with* $\Delta C(t) = 0$. *It follows that we have found all indefinite sums of* 6^t, *and we write*

$$\sum 6^t = \frac{6^t}{5} + C(t),$$

where $C(t)$ *is any function with the same domain as* 6^t *and* $\Delta C(t) = 0$.

Theorem 1.2.3. *If* $z(t)$ *is an indefinite sum of* $y(t)$, *then every indefinite sum of* $y(t)$ *is given by*

$$\sum y(t) = z(t) + C(t),$$

where $C(t)$ *has the same domain as* y *and* $\Delta C(t) = 0$.

Proof. Given that $z(t) = \sum y(t)$.

Let us assume that $C(t)$ has the same domain as $y(t)$ so that $\Delta C(t) = 0$.

Then

$$\begin{aligned} \Delta\big(z(t) + C(t)\big) &= \Delta z(t) + \Delta C(t) \\ &= \Delta z(t) + 0 \\ &= \Delta\big(\sum y(t)\big) \\ &= y(t). \end{aligned}$$

Thus $z(t) + C(t)$ is an indefinite sum of $y(t)$.

Further, if $f(t)$ is any indefinite sum of $y(t)$, then

$$\begin{aligned} \Delta\big(f(t) - z(t)\big) &= \Delta f(t) - \Delta z(t) \\ &= y(t) - y(t) \\ &= 0, \end{aligned}$$

so $f(t) = z(t) + C(t)$ with $\Delta C(t) = 0$. Thus, every indefinite sum of $y(t)$ is given by

$$\sum y(t) = z(t) + C(t),$$

where $C(t)$ is any function with the same domain as $y(t)$ and $\Delta C(t) = 0$. □

Corollary 1.2.4. *Let $y(t)$ be defined on a set of the type $\{a, a+1, a+2, \cdots\}$, where a is any real number, and let $z(t)$ be an indefinite sum of $y(t)$. Then every indefinite sum of $y(t)$ is given by*

$$\sum y(t) = z(t) + C,$$

where C is an arbitrary constant.

Theorem 1.2.5. *Let 'a' be a constant. Then, for $\Delta C(t) = 0$,*

(a) $\sum a^t = \frac{a^t}{a-1} + C(t), \quad (a \neq 1).$

(b) $\sum \sin at = -\frac{\cos a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + C(t), \quad (a \neq 2n\pi).$

(c) $\sum \cos at = -\frac{\sin a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + C(t), \quad (a \neq 2n\pi).$

(d) $\sum \log t = \log\Gamma(t) + C(t), \quad (t > 0).$

(e) $\sum t^{\underline{a}} = \frac{t^{\underline{a+1}}}{a+1} + C(t), \quad (a \neq -1).$

(f) $\sum \binom{t}{a} = \binom{t}{a+1} + C(t).$

(g) $\sum \binom{a+t}{t} = \binom{a+t}{t-1} + C(t).$

Proof. (a) We know that

$$\Delta a^t = (a-1)a^t.$$

Then,

$$\Delta\Big(\frac{a^t}{a-1}\Big) \quad = \quad a^t.$$

Since $\frac{a^t}{a-1}$ is an indefinite sum of a^t, we can write,

$$\sum a^t = \frac{a^t}{a-1} + C(t),$$

with $\Delta C(t) = 0$.

(b) We know that

$$\Delta \cos aT = -2\sin\frac{a}{2}\sin a\bigg(T + \frac{1}{2}\bigg).$$

Put $T = t - \frac{1}{2}$. Then,

$$\frac{\Delta \cos a\big(t - \frac{1}{2}\big)}{-2\sin\frac{a}{2}} = \sin at.$$

Since $\frac{\cos a\big(t-\frac{1}{2}\big)}{-2\sin\frac{a}{2}}$ is an indefinite sum of $\sin at$, we can write

$$\sum \sin at = -\frac{\cos a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + C(t), \quad (a \neq 2n\pi)$$

with $\Delta C(t) = 0$.

(c) We know that

$$\Delta \sin aT \quad = \quad 2\cos a\big(T + \frac{1}{2}\big)\sin\frac{a}{2}.$$

Put $T = t - \frac{1}{2}$. Then,

$$\Delta\frac{\sin a\big(t-\frac{1}{2}\big)}{2\sin\frac{a}{2}} \quad = \quad \cos at.$$

Since $\frac{\sin a\big(t-\frac{1}{2}\big)}{2\sin\frac{a}{2}}$ is an indefinite sum of $\cos at$, we can write

$$\sum \cos at = -\frac{\sin a(t-\frac{1}{2})}{2\sin\frac{a}{2}} + C(t), \quad (a \neq 2n\pi)$$

with $\Delta C(t) = 0$.

(d) We know that

$$\Delta \log\Gamma(t) = \log(t).$$

Since $\log\Gamma(t)$ is an indefinite sum of $\log t$, we can write

$$\sum \log t = \log\Gamma(t) + C(t), \quad (t > 0)$$

with $\Delta C(t) = 0$.

(e) We know that $\Delta_t t^{\underline{r}} = r t^{\underline{r-1}}$.

Then,

$$\begin{aligned} \Delta t^{\underline{a+1}} &= (a+1)t^{\underline{a+1-1}} \\ \Longrightarrow \Delta \frac{t^{(a+1)}}{a+1} &= t^{\underline{a}}. \end{aligned}$$

Since $\frac{t^{\underline{a+1}}}{a+1}$ is an indefinite sum of $t^{\underline{a}}$, we can write

$$\sum t^{\underline{a}} = \frac{t^{\underline{a+1}}}{a+1} + C(t), \quad (a \neq -1)$$

with $\Delta C(t) = 0$.

(f) We know that $\Delta_t \binom{t}{r} = \binom{t}{r-1} \quad (r \neq 0)$.

Then,

$$\Delta \binom{t}{a+1} = \binom{t}{a}.$$

Since $\binom{t}{a+1}$ is an indefinite sum of $\binom{t}{a}$, we can write

$$\sum \binom{t}{a} = \binom{t}{a+1} + C(t),$$

with $\Delta C(t) = 0$.

(g) We know that $\Delta_t \binom{r+t}{t} = \binom{r+t}{t+1}$.

Then,

$$\Delta \binom{a+t}{t-1} = \binom{a+t}{t}.$$

Since $\binom{a+t}{t-1}$ is an indefinite sum of $\binom{a+t}{t}$, we can write

$$\sum \binom{a+t}{t} = \binom{a+t}{t-1} + C(t)$$

with $\Delta C(t) = 0$. □

Note: All the formulas in the above Theorem remain valid if a constant shift is introduced in the 't' variable.

Example 1.2.6. *Find the solution of*

$$y(t+2) - 2y(t+1) + y(t) = t^{\underline{2}}, \qquad (t = 0, 1, 2, ...),$$

so that $y(0) = -1, y(1) = 3.$

Since $\Delta^2 y(t) = t^{\underline{2}}$*, we have* $\Delta(\Delta y(t)) = t^{\underline{2}}.$

$\Longrightarrow \Delta y(t)$ *is an indefinite sum of* $t^{\underline{2}}.$

Since $\sum t^{\underline{a}} = \frac{t^{\underline{a+1}}}{a+1} + C(t)$, *we have*

$$\frac{t^{\underline{3}}}{3} + C \;=\; \Delta y(t),$$

and hence

$$\begin{aligned} y(t) &= \sum \frac{t^{\underline{3}}}{3} + C \sum t^{\underline{0}} + D \\ &= \frac{t^{\underline{4}}}{12} + Ct + D, \end{aligned}$$

where C *and* D *are constants. Using the values of* y *at* $t = 0$ *and* $t = 1$, *we get* $D = -1$ *and* $C = 4$ *, so the unique solution is*

$$y(t) = \frac{t^{\underline{4}}}{12} + 4t - 1.$$

Theorem 1.2.7. *The following are some general properties of indefinite sums:*

(a) $\sum(y(t) + z(t)) = \sum y(t) + \sum z(t).$

(b) $\sum Dy(t) = D \sum y(t)$ *if D is constant.*

(c) $\sum(y(t)\Delta z(t)) = y(t)z(t) - \sum Ez(t)\Delta y(t).$

(d) $\sum(Ey(t)\Delta z(t)) = y(t)z(t) - \sum z(t)\Delta y(t).$

Proof. (a) By the definition of indefinite sum,

$$\begin{aligned} \Delta\left(\sum y(t) + \sum z(t)\right) &= \Delta\left(\sum y(t)\right) + \Delta\left(\sum z(t)\right) \\ &= y(t) + z(t). \\ \Longrightarrow \quad \sum\left(y(t) + z(t)\right) &= \sum y(t) + \sum z(t). \end{aligned}$$

(b) By the definition of indefinite sum,

$$\begin{aligned}\Delta\left(D\sum y(t)\right) &= D\left(\Delta\left(\sum y(t)\right)\right)\\ &= Dy(t).\\ \implies \quad \sum Dy(t) = D\sum y(t).\end{aligned}$$

(c) We know that

$$\Delta(y(t)z(t)) = y(t)\Delta z(t) + Ez(t)\Delta y(t).$$

$$\begin{aligned}\implies \sum\Big(y(t)\Delta z(t) + Ez(t)\Delta y(t)\Big) &= y(t)z(t)\\ \implies \sum\Big(y(t)\Delta z(t)\Big) + \sum\Big(Ez(t)\Delta y(t)\Big) &= y(t)z(t) \qquad \textbf{[by (a)]}\\ \implies \sum\Big(y(t)\Delta z(t)\Big) &= y(t)z(t) - \sum\Big(Ez(t)\Delta y(t)\Big).\end{aligned}$$

(d) We know that

$$\Delta\Big(z(t)y(t)\Big) = z(t)\Delta y(t) + Ey(t)\Delta z(t).$$

$$\begin{aligned}\implies \sum\Big(z(t)\Delta y(t) + Ey(t)\Delta z(t)\Big) &= z(t)y(t)\\ \implies \sum\Big(z(t)\Delta y(t)\Big) + \sum\Big(Ey(t)\Delta z(t)\Big) &= z(t)y(t) \qquad \textbf{[by (a)]}\\ \implies \sum\Big(Ey(t)\Delta z(t)\Big) &= y(t)z(t) - \sum z(t)\Delta y(t).\end{aligned}$$

□

Remark: Parts (c) and (d) of Theorem 1.2.7. are known as "summation by parts" formula.

Example 1.2.8. *Compute* $\sum ta^t$ $(a \neq 1)$.

We know that

$$\sum(y(t)\Delta z(t)) = y(t)z(t) - \sum(Ez(t)\Delta y(t)).$$

If we choose $y(t) = t$ and $\Delta z(t) = a^t$, then we get $z(t) = \frac{a^t}{a-1}$.

So, we have

$$\begin{aligned}\sum ta^t &= t\Big(\frac{a^t}{a-1}\Big) - \sum \frac{a^{t+1}}{a-1}\Delta t + C(t) \\ &= \frac{ta^t}{a-1} - \frac{a}{a-1}\sum a^t + C(t) \\ &= \frac{ta^t}{a-1} - \frac{a}{(a-1)^2}a^t + C(t),\end{aligned}$$

where $\Delta C(t) = 0$.

Example 1.2.9. *Compute* $\sum \binom{t}{5}\binom{t}{2}$.

Consider the summation by parts formula

$$\sum(y(t)\Delta z(t)) = y(t)z(t) - \sum(Ez(t)\Delta y(t)).$$

By taking $y(t) = \binom{t}{2}$ *and* $\Delta z(t) = \binom{t}{5}$*, we get* $z(t) = \binom{t}{6}$*, and hence*

$$\sum \binom{t}{5}\binom{t}{2} = \binom{t}{6}\binom{t}{2} - \sum \binom{t+1}{6}\binom{t}{1} + C(t).$$

Now, we applying summation by parts to the last sum with $y_1(t) = \binom{t}{1}, \Delta z_1(t) = \binom{t+1}{6}$*, and* $z_1(t) = \binom{t+1}{7}$*, we have*

$$\begin{aligned}\sum \binom{t}{5}\binom{t}{2} &= \binom{t}{6}\binom{t}{2} - \left[\binom{t+1}{7}\binom{t}{1} - \sum \binom{t+2}{7}\right] + C(t) \\ &= \binom{t}{6}\binom{t}{2} - \binom{t+1}{7} + \binom{t+2}{8} + C(t),\end{aligned}$$

where $\Delta C(t) = 0$.

Note:

For the remainder of this section, we will assume that the domain of $y(t)$ is the natural numbers $N = \{1, 2, 3, \ldots\}$. Sequence notation will be used for the function $y(t)$: That is,

$$y(t) \leftrightarrow \{y_n\},$$

where $n \in \mathbb{N}$. It will be convenient to use the convention

$$\sum_{k=a}^{b} y_k = 0$$

whenever $a > b$.

Note:

Observe that for m fixed and $n \geq m$,

$$\Delta_n \left(\sum_{k=m}^{n-1} y_k \right) = y_n,$$

and for p fixed and $p \geq n$,

$$\Delta_n \left(\sum_{k=n}^{p} y_k \right) = -y_n.$$

Corollary 1.2.10. ***(Relation between definite and indefinite sums)***

From the above note

$$\sum y_n = \sum_{k=m}^{n-1} y_k + C \qquad (m \leq n) \tag{1.2}$$

for some constant C and, alternatively, that

$$\sum y_n = -\sum_{k=n}^{p} y_k + D \qquad (p \geq n) \tag{1.3}$$

for some constant D. Equations (1.2) and (1.3) give us a way of relating indefinite sums to definite sums.

Example 1.2.11. *Compute the definite sum $\sum_{k=1}^{n-1} (\frac{2}{3})^k$.*

By equation (1.2) and $\sum a^t = \frac{a^t}{a-1} + C(t)$, we have

$$\begin{aligned} \sum_{k=1}^{n-1} (\frac{2}{3})^k &= \sum (\frac{2}{3})^n + C \\ &= \frac{(\frac{2}{3}^n)}{\frac{2}{3} - 1} + C \\ &= -3(\frac{2}{3})^n + C, \qquad (n = 2, 3, ...). \end{aligned}$$

To evaluate C, let $n = 2$. Then, we get

$$\begin{aligned} \frac{2}{3} &= -3(\frac{2}{3})^2 + C, \\ \Rightarrow C &= 2, \end{aligned}$$

and so

$$\sum_{k=1}^{n-1} (\frac{2}{3})^k = 2 - 3(\frac{2}{3})^n \qquad (n = 2, 3, ...).$$

Theorem 1.2.12. *If z_n is an indefinite sum of y_n, then*

$$\sum_{k=m}^{n-1} y_k = [z_k]_m^n = z_n - z_m.$$

Proof. Since z_n is an indefinite sum of y_n, we have

$$\sum y_n = z_n.$$

and

$$\begin{aligned} y_n &= \Delta z_n \\ &= z_{n+1} - z_n. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{k=m}^{n-1} y_k &= \sum_{k=m}^{n-1} (z_{k+1} - z_k) \\ &= z_{m+1} - z_m + z_{m+2} - z_{m+1} + z_{m+3} - z_{m+2} + \dots\dots + z_n - z_{n-1} \\ &= z_n - z_m. \end{aligned}$$

□

Example 1.2.13. *Compute* $\sum\limits_{k=1}^{l} k^2$.

Recall that $k^{\underline{1}} = k$ *and* $k^{\underline{2}} = k(k-1)$.

Then $k^2 = k^{\underline{1}} + k^{\underline{2}}$, *and so,*

$$\begin{aligned} \sum k^2 &= \sum k^{\underline{1}} + \sum k^{\underline{2}} \\ &= \frac{k^{\underline{2}}}{2} + \frac{k^{\underline{3}}}{3} + C \end{aligned}$$

by Theorem 1.2.5(e). From Theorem 1.2.12, we have

$$\begin{aligned} \sum_{k=1}^{l} k^2 &= \left[\frac{k^{\underline{2}}}{2} + \frac{k^{\underline{3}}}{3}\right]_1^{l+1} \\ &= \frac{(l+1)^{\underline{2}}}{2} + \frac{(l+1)^{\underline{3}}}{3} - \frac{1^{\underline{2}}}{2} - \frac{1^{\underline{3}}}{3} \\ &= \frac{(l+1)l}{2} + \frac{(l+1)l(l-1)}{3} \\ &= \frac{(l+1)l}{2} + \frac{(l+1)l(l-1)}{3} \\ &= \frac{l(l+1)(2l+1)}{6}. \end{aligned}$$

The next theorem gives a version of the summation by parts method for definite sums.

Theorem 1.2.14. *If $m < n$, then*

$$\sum_{k=m}^{n-1} a_k \Delta b_k = [a_k b_k]_m^n - \sum_{k=m}^{n-1} (\Delta a_k)\, b_{k+1}.$$

Proof. Choosing $y(n) = a_n$ and $z(n) = b_n$ in Theorem 1.2.7(c), we have

$$\sum a_n \Delta b_n = a_n b_n - \sum (\Delta a_n)\, b_{n+1}.$$

From equation (1.2), we have

$$\sum_{k=m}^{n-1} a_k \Delta b_k = a_n b_n - \sum_{k=m}^{n-1} (\Delta a_k)\, b_{k+1} + C.$$

With $n = m + 1$, the preceding equation becomes

$$a_m \Delta b_m = a_{m+1} b_{m+1} - (\Delta a_m)\, b_{m+1} + C.$$

It follows that $C = -a_m b_m$, and the proof is complete. □

Remark 1.2.15. *An equivalent form of Theorem 1.2.14 is Abel's summation formula:*

$$\sum_{k=m}^{n-1} c_k d_k = d_n \sum_{k=m}^{n-1} c_k - \sum_{k=m}^{n-1} \left(\sum_{i=m}^{k} c_i \right) \Delta d_k.$$

Example 1.2.16. *Compute $\sum_{k=1}^{n-1} k3^k$.*

By Theorem 1.2.14 with $a_k = k$ and $\Delta b_k = 3^k$,

$$\sum_{k=1}^{n-1} k3^k = \left[k \frac{3^k}{2} \right]_1^n - \sum_{k=1}^{n-1} \frac{3^{k+1}}{2}.$$

From Theorem 1.2.12 and Theorem 1.2.5(a),

$$\sum_{k=1}^{n-1} 3^k = \frac{3^n - 3}{2}.$$

Returning to our calculation, we have

$$\sum_{k=1}^{n-1} k3^k = \frac{n3^n - 3}{2} - \frac{3}{2}\left(\frac{3^n - 3}{2}\right)$$
$$= \frac{(2n-3)3^n + 3}{4}.$$

Note: The methods used in Example 1.2.16. allow us to compute any definite sum of sequences of the form $p(n)a^n, p(n)\sin$ an, $p(n)\cos an$, and $p(n)\binom{n}{a}$, where $p(n)$ is a polynomial in n. However, we must have as many repetitions of summation by parts as the degree of p.

There is a special method of summation that is based on Eq. (1.1) for the n^{th} difference of a function:

$$\Delta^n y(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} y(n-k)$$
$$= \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} y(i),$$

where we have used the change of index $i = n - k$ and the fact that $\binom{n}{n-i} = \binom{n}{i}$. It follows that

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} y(i) = (-1)^n \Delta^n y(0). \tag{1.4}$$

Example 1.2.17. *Compute* $\sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{i+a}{m}$.

Let $y(i) = \binom{i+a}{m}$ *in Eq.* (1.4).

From Theorem 1.1.12*(b),* $\Delta^n \binom{i+a}{m} = \binom{i+a}{m-n}$.

Thus, by Eq. (1.4)*, we get*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{i+a}{m} = (-1)^n \binom{a}{m-n}. \tag{1.5}$$

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Definition of indefinite sum
2. Indefinite sum of some basic functions
3. General properties of indefinite sums

4. Relation between indefinite sums and definite sums
5. Abel's summation formula

Check your Progress:

1. For a constant a, $\sum \binom{t}{a} =$ ——?

   (A) $\binom{1+t}{a-1}$ (B) $\binom{1+t}{a+1}$ (C) $\binom{t}{a+1}$ (D) $\binom{t}{a-1}$

2. If C is a constant, then $\sum Cy(t) =$ ——?

   (A) $\sum y(t)$ (B) $C \sum y(t)$ (C) $\sum y(Ct)$ (D) None of these

3. If z_n is an indefinite sum of y_n, then $\sum_{k=m}^{n-1} y_k =$ ....?

   (A) $z_n - z_m$ (B) $z_{n-1} - z_m$ (C) $y_n - y_m$ (D) $y_{n-1} - y_m$

1.3 Generating Functions and Approximate Summation

In Section 1.2, we discussed a number of methods by which finite sums can be computed. However, most sums, like most integrals, cannot be expressed in terms of the elementary functions of calculus. There are functions such as $y(t) = \frac{1}{t}$ that can be integrated exactly,

$$\int_a^b \frac{1}{t} dt = \log \frac{b}{a}, \quad (b > a > 0),$$

but for which there is no elementary formula for the corresponding sum:

$$\sum_{k=1}^{n} \frac{1}{k}.$$

The main result of this section, called the Euler summation formula, will give us a technique for approximating a sum if the corresponding integral can be computed. To formulate this result, we will use a generating function, which is itself important in the analysis of difference equations, and a family of special functions called the Bernoulli polynomials.

Definition 1.3.1. *Let $\{y_k(t)\}$ be a sequence of (possibly constant) functions.*

(a) If there is a function $g(t,x)$ so that

$$g(t,x)=\sum_{k=0}^{\infty} y_k(t)x^k$$

for all x in an open interval about zero, then g is called the "generating function" for $\{y_k(t)\}$.

(b) If there is a function $h(t,x)$ so that

$$h(t,x)=\sum_{k=0}^{\infty} \frac{y_k(t)x^k}{k!}$$

for all x in an open interval about zero, then h is called the "exponential generating function" for $\{y_k(t)\}$.

Note that for each $t, y_k(t)$ is the k^{th} coefficient in the power series for $g(t,x)$ with respect to x at $x=0$.

Example 1.3.2. *Let $y_k(t)=(f(t))^k$ for some function $f(t)$. Then*

$$\begin{aligned} g(t,x) &= \sum_{k=0}^{\infty}(f(t))^k x^k \\ &= \sum_{k=0}^{\infty}(f(t)x)^k \\ &= \frac{1}{1-f(t)x} \text{ if } |f(t)x|<1. \end{aligned}$$

$\Longrightarrow \quad \frac{1}{1-f(t)x}$ *is the generating function for the sequence $\{y_k(t)\}$.*

Next,

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{1}{1-f(t)x}\right) &= \frac{\partial}{\partial x}\sum_{k=0}^{\infty}(f(t))^k x^k \\ \Longrightarrow \frac{f(t)}{(1-f(t)x)^2} &= \sum_{k=0}^{\infty} k(f(t))^k x^{k-1} \\ \Longrightarrow \frac{xf(t)}{(1-f(t)x)^2} &= \sum_{k=0}^{\infty} k(f(t))^k x^k. \end{aligned}$$

So, $\frac{xf(t)}{(1-f(t)x)^2}$ is the generating function for the sequence $\left\{k(f(t))^k\right\}$.

Definition 1.3.3. *The "Bernoulli polynomials" $B_k(t)$ are defined by the equation*

$$\frac{xe^{tx}}{e^x-1}=\sum_{k=0}^{\infty}\frac{B_k(t)}{k!}x^k.$$

In other words, $\frac{xe^{tx}}{e^x-1}$ is the exponential generating function for the sequence $B_k(t)$.

Definition 1.3.4. *The "Bernoulli numbers" B_k are given by $B_k = B_k(0)$, the value of the k^{th} Bernoulli polynomial at $t = 0$.*

Remark 1.3.5. *Compute the first four Bernoulli numbers.*
Consider

$$\frac{xe^{tx}}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} x^k.$$
$$\implies e^{tx} = \frac{e^x - 1}{x} \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} x^k.$$

Then expanding the exponential functions on each side in their Taylor series about zero and collecting terms containing the same power of x, we get

$$\begin{aligned}
&1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \cdots \\
&= \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \cdots\right)\left(B_0(t) + \frac{B_1(t)}{1!}x + \frac{B_2(t)}{2!}x^2 + \cdots\right) \\
&= B_0(t) + \left(\frac{B_1(t)}{1!} + \frac{B_0(t)}{2!}\right)x + \left(\frac{B_2(t)}{2!} + \frac{B_1(t)}{2!1!} + \frac{B_0(t)}{3!}\right)x^2 + \cdots .
\end{aligned}$$

Equating coefficients of like powers of x, we have

$$B_0(t) = 1, \quad B_1(t) + \tfrac{B_0(t)}{2} = t, \quad \tfrac{B_2(t)}{2} + \tfrac{B_1(t)}{2} + \tfrac{B_0(t)}{6} = \tfrac{t^2}{2} \cdots .$$

Thus, the first few Bernoulli polynomials are given by

$$B_0(t) = 1, \quad B_1(t) = t - \frac{1}{2}, \quad B_2(t) = t^2 - t + \frac{1}{6}, \qquad (1.6)$$
$$B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t, \cdots$$

Then, the first four Bernoulli numbers are given by

$$\begin{aligned}
B_0 &= B_0(0) = 1, \\
B_1 &= B_1(0) = \frac{-1}{2}, \\
B_2 &= B_2(0) = \frac{1}{6}, \\
B_3 &= B_3(0) = 0.
\end{aligned}$$

$$\therefore B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0. \qquad (1.7)$$

Theorem 1.3.6. *(Properties of Bernoulli polynomials)*

(a) $B'_k(t) = kB_{k-1}(t), \quad (k \geq 1)$.

(b) $\Delta_k B_k(t) = kt^{k-1}, \quad (k \geq 0)$.

(c) $B_k = B_k(0) = B_k(1), \quad (k \neq 1)$.

(d) $B_{2m+1} = 0, \quad (m \geq 1)$.

Proof. (a) We know that

$$\frac{xe^{tx}}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} x^k.$$

Differentiating with respect to t on both sides, we get

$$\begin{aligned} \implies \frac{x^2 e^{tx}}{e^x - 1} &= \sum_{k=0}^{\infty} \frac{B'_k(t)}{k!} x^k \\ \implies x\Big(\sum_{k=0}^{\infty} \frac{B_k(t)}{k!} x^k\Big) &= \sum_{k=0}^{\infty} \frac{B'_k(t)}{k!} x^k \\ \implies \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} x^{k+1} &= \sum_{k=0}^{\infty} \frac{B'_k(t)}{k!} x^k. \end{aligned}$$

Now, making the change of index $k \to k-1$ in the left-hand sum, we get

$$\sum_{k=1}^{\infty} \frac{B_{k-1}(t)}{(k-1)!} x^k = \sum_{k=0}^{\infty} \frac{B'_k(t)}{k!} x^k.$$

Equating the coefficients of x^k, we get

$$\begin{aligned} \frac{B_{k-1}(t)}{(k-1)!} &= \frac{B'_k(t)}{k!} \\ \implies B'_k(t) &= kB_{k-1}(t), \ k \geq 1. \end{aligned}$$

(b) Consider

$$\frac{xe^{tx}}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k(t)}{k!} x^k.$$

Next, taking the difference of both sides, we get

$$\begin{aligned}
\sum_{k=0}^{\infty} \frac{\Delta_t B_k(t)}{k!} x^k &= \frac{x}{e^x - 1} \Delta_t e^{tx} \\
&= \frac{x}{e^x - 1} \left(e^{(t+1)x} - e^{tx} \right) \\
&= \frac{x}{e^x - 1} e^{tx} (e^x - 1) \\
&= x e^{tx} \\
&= x \sum_{k=0}^{\infty} \frac{t^k x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{t^k x^{k+1}}{k!} \\
&= \sum_{k=1}^{\infty} \frac{t^{k-1} x^k}{(k-1)!}.
\end{aligned}$$

Equating the coefficients, we get

$$\begin{aligned}
\frac{\Delta_t B_k(t)}{k!} &= \frac{t^{k-1}}{(k-1)!} \\
\implies \Delta_t B_k(t) &= k t^{k-1}, \ k \geq 0.
\end{aligned}$$

(c) Consider

$$\begin{aligned}
\Delta_t B_k(t) &= k t^{k-1} \\
\implies B_k(t+1) - B_k(t) &= k t^{k-1}.
\end{aligned}$$

Putting $t = 0$, we get

$$B_k(1) - B_k(0) = 0.$$

$$\implies B_k = B_k(0) = B_k(1), \ k \neq 1.$$

□

Corollary 1.3.7. *If* $k = 0, 1, 2, \cdots$*, then*

$$\sum t^k = \frac{1}{k+1} B_{k+1}(t) + C(t),$$

where $\Delta C(t) = 0$*.*

Proof. We know that

$$\begin{aligned}
\Delta_t B_k(t) &= kt^{k-1},\ k \geq 0 \\
\Longrightarrow \sum kt^{k-1} &= B_k(t) + C(t),\ \text{where } \Delta C(t) = 0 \\
\Longrightarrow \sum k+1t^k &= B_{k+1}(t) + C(t) \\
\Longrightarrow \sum t^k &= \frac{1}{k+1} B_{k+1}(t) + C(t).
\end{aligned}$$

□

Theorem 1.3.8. ***(Euler summation formula)*** *Suppose that the $2m^{th}$ derivative of $y(t), y^{(2m)}(t)$, is continuous on $[1,n]$ for some integers $m \geq 1$ and $n \geq 2$. Then*

$$\sum_{k=1}^{n} y(k) = \int_1^n y(t)dt + \frac{y(n)+y(1)}{2} + \sum_{i=1}^{m} \frac{B_{2i}}{(2i)!} \left[y^{(2i-1)}(n) - y^{(2i-1)}(1) \right]$$
$$- \frac{1}{(2m)!} \int_1^n y^{(2m)}(t) B_{2m}(t - \lfloor t \rfloor) dt,$$

where $\lfloor t \rfloor$ = the greatest integer less than or equal to t (called the "floor function" or the "greatest integer function").

Proof. We know that

$$B_1(t) = t - \frac{1}{2}.$$

Then,

$$B_1(t - \lfloor t \rfloor) = t - \lfloor t \rfloor - \frac{1}{2}.$$

For each k,

$$\begin{aligned}
\int_k^{k+1} B_1(t - \lfloor t \rfloor) y'(t) dt &= \int_k^{k+1} \left(t - \lfloor t \rfloor - \frac{1}{2} \right) y'(t) dt \\
&= \int_k^{k+1} \left(t - k - \frac{1}{2} \right) y'(t) dt \\
&= \left(t - k - \frac{1}{2} \right) y(t) \Big|_k^{k+1} - \int_k^{k+1} y(t) dt \\
&= \frac{y(k+1)}{2} + \frac{y(k)}{2} - \int_k^{k+1} y(t) dt \\
&= \frac{y(k+1) + y(k)}{2} - \int_k^{k+1} y(t) dt.
\end{aligned}$$

That is,

$$\int_k^{k+1} B_1(t - \lfloor t \rfloor) y'(t) dt = \frac{y(k+1) + y(k)}{2} - \int_k^{k+1} y(t) dt. \tag{1.8}$$

Similarly, for $i = 1, 2, \dots, 2m-1$,

$$\begin{aligned}
\int_k^{k+1} B_i(t - \lfloor t \rfloor) y^{(i)}(t) dt &= \int_k^{k+1} B_i(t-k) y^{(i)}(t) dt \\
&= y^{(i)}(t) \frac{B_{i+1}(t-k)}{i+1}\bigg|_k^{k+1} - \int_k^{k+1} \frac{B_{i+1}(t-k)}{i+1} y^{(i+1)}(t) dt \\
&= \frac{B_{i+1}}{i+1}\left\{ y^{(i)}(k+1) - y^{(i)}(k) \right\} - \frac{1}{i+1} \int_k^{k+1} B_{i+1}(t - \lfloor t \rfloor) y^{(i+1)}(t) dt. \quad (1.9)
\end{aligned}$$

Summing equation (1.8) as k goes from 1 to $n-1$, we have

$$\begin{aligned}
\int_1^n B_1(t - \lfloor t \rfloor) y'(t) dt &= \sum_{k=1}^{n-1} \frac{y(k) + y(k+1)}{2} - \int_1^n y(t) dt \\
&= \frac{y(1) + y(2)}{2} + \frac{y(2) + y(3)}{2} + \dots \\
&\quad + \frac{y(n-1) + y(n)}{2} - \int_1^n y(t) dt. \\
&= \frac{y(1)}{2} + y(2) + y(3) + \dots + y(n-1) \\
&\quad + \frac{y(n)}{2} - \int_1^n y(t) dt \\
&= \left(\sum_{k=1}^{n} y(k) \right) - \left(\frac{y(1) + y(n)}{2} \right) - \int_1^n y(t) dt. \quad (1.10)
\end{aligned}$$

Summing equation (1.9) as k goes from 1 to $n-1$, we have

$$\begin{aligned}
\int_1^n B_i(t - \lfloor t \rfloor) y^{(i)}(t) dt &= \frac{B_{i+1}}{i+1} \sum_{k=1}^{n-1} \left[y^{(i)}(k+1) - y^{(i)}(k) \right] - \frac{1}{i+1} \int_1^n B_{i+1}(t - \lfloor t \rfloor) y^{(i+1)}(t) dt \\
&= \frac{B_{i+1}}{i+1} \left[y^{(i)}(2) - y^{(i)}(1) + y^{(i)}(3) - y^{(i)}(2) + \dots + y^{(i)}(n) - y^{(i)}(n-1) \right] \\
&\quad - \frac{1}{i+1} \int_1^n B_{i+1}(t - \lfloor t \rfloor) y^{(i+1)}(t) dt \\
&= \frac{B_{i+1}}{i+1} \left[y^{(i)}(n) - y^{(i)}(1) \right] - \frac{1}{i+1} \int_1^n B_{i+1}(t - \lfloor t \rfloor) y^{(i+1)}(t) dt. \quad (1.11)
\end{aligned}$$

Now from (1.10),

$$
\begin{aligned}
& \sum_{k=1}^{n} y(k) - \frac{1}{2}(y(1)+y(n)) - \int_1^n y(t)dt \\
= & \int_1^n B_1(t-\lfloor t \rfloor) y'(t)dt \\
= & \frac{B_2}{2}[y'(n)-y'(1)] - \frac{1}{2}\int_1^n B_2(t-\lfloor t \rfloor) y^{(2)}(t)dt \qquad \text{(by (1.11))} \\
= & \frac{B_2}{2}[y'(n)-y'(1)] - \frac{1}{2}\left\{\frac{B_3}{3}[y^{(2)}(n)-y^{(2)}(1)] - \frac{1}{3}\int_1^n B_3(t-\lfloor t \rfloor) y^{(3)}(t)dt\right\} \\
= & \frac{B_2}{2}[y'(n)-y'(1)] + \frac{1}{2.3}\left\{\frac{B_4}{4}[y^{(3)}(n)-y^{(3)}(1)] - \frac{1}{4}\int_1^n B_4(t-\lfloor t \rfloor) y^{(4)}(t)dt\right\} \\
= & \sum_{i=1}^{m} \frac{B_{2i}}{(2i)!}\left[y^{(2i-1)}(n) - y^{(2i-1)}(1)\right] - \frac{1}{(2m)!}\int_1^n B_{2m}(t-\lfloor t \rfloor) y^{(2m)}(t)dt.
\end{aligned}
$$

Hence the proof. □

Example 1.3.9. *Approximate* $\sum_{k=1}^{n} k^{\frac{1}{2}}$.

Put $y(t) = t^{\frac{1}{2}}$ and $m = 1$ in the Euler summation formula. Then,

$$
\begin{aligned}
\sum_{k=1}^{n} k^{1/2} &= \int_1^n t^{1/2}dt + \frac{n^{1/2}+1}{2} + \frac{B_2}{2!}[y'(n)-y'(1)] - \frac{1}{2!}\int_1^n y''(t)B_2(t-\lfloor t \rfloor)dt \\
&= \int_1^n t^{1/2}dt + \frac{n^{1/2}+1}{2} + \frac{1}{12}\left[\frac{1}{2}n^{-1/2} - 1/2\right] - \frac{1}{2!}\int_1^n \left(-\frac{1}{4}t^{-3/2}\right) B_2(t-\lfloor t \rfloor)dt \\
&= \frac{2}{3}n^{\frac{3}{2}} + \frac{n^{1/2}}{2} + \frac{1}{2} + \frac{1}{24}n^{\frac{-1}{2}} - \frac{1}{24} + \frac{1}{8}\int_1^n t^{\frac{-3}{2}} B_2(t-\lfloor t \rfloor)dt \\
&= \frac{2}{3}n^{\frac{3}{2}} + \frac{1}{2}n^{\frac{1}{2}} + \frac{1}{24}n^{\frac{-1}{2}} - \frac{5}{24} + \frac{1}{8}\int_1^n t^{-3/2} B_2(t-\lfloor t \rfloor)dt. \qquad (1.12)
\end{aligned}
$$

Now,

$$
B_2(x) = x^2 - x + \frac{1}{6} \implies B_2'(x) = 2x - 1 \text{ and } B_2''(x) = 2.
$$

Then,

$$
\begin{aligned}
B_2'(x) = 0 &\implies x = \frac{1}{2} \\
&\implies B_2''\left(\frac{1}{2}\right) = 2 > 0.
\end{aligned}
$$

$\implies$ $B_2(x)$ has a minimum value at $x = 1/2$, and

$B_2\left(\frac{1}{2}\right) = \frac{1}{4} - \frac{1}{2} + \frac{1}{6} = \frac{-1}{12}$.

$\therefore$ min $B_2(x) = \frac{-1}{12}$.

Also, $B_2(x) \le \frac{1}{6}$ for $0 \le x \le 1$.

$$\therefore -\frac{1}{12} \le B_2(x) \le \frac{1}{6} \qquad \text{for } 0 \le x \le 1.$$

Since, $0 \le t - [t] < 1 \quad \forall t$, we have

$$\begin{aligned}
\frac{1}{8}\int_1^n t^{-3/2}\left(\frac{-1}{12}\right)dt &\le \frac{1}{8}\int_1^n t^{-3/2}B_2(t-\lfloor t\rfloor)dt \leqslant \frac{1}{8}\int_1^n t^{-3/2}(1/6)dt\\
\Rightarrow -\frac{1}{96}\int_1^n t^{-3/2}dt &\le \frac{1}{8}\int_1^n t^{-3/2}B_2(t-\lfloor t\rfloor)dt \leqslant \frac{1}{48}\int_1^n t^{-3/2}dt\\
\Rightarrow -\frac{1}{96}\left[-2t^{-1/2}\right]_1^n &\le \frac{1}{8}\int_1^n t^{-3/2}B_2(t-\lfloor t\rfloor)dt \le \frac{1}{48}\left[-2t^{-1/2}\right]_1^n\\
\Rightarrow -\frac{1}{48}\left[1-n^{-1/2}\right] &\le \frac{1}{8}\int_1^n t^{-3/2}B_2(t-\lfloor t\rfloor)dt \le \frac{1}{24}\left[1-n^{-1/2}\right].
\end{aligned}$$

Thus, by (1.12), we have

$$\begin{aligned}
&\frac{2}{3}n^{3/2}+\frac{1}{2}n^{1/2}+\frac{1}{24}n^{-1/2}-\frac{5}{24}-\frac{1}{48}\left(1-n^{-1/2}\right)\\
&\quad\le \sum_{k=1}^{n}k^{1/2} \le \frac{2}{3}n^{3/2}+\frac{1}{2}n^{1/2}+\frac{1}{24}n^{-1/2}-\frac{5}{24}+\frac{1}{24}\left(1-n^{-1/2}\right)\\
\Rightarrow &\qquad \frac{2}{3}n^{\frac{3}{2}}+\frac{1}{2}n^{\frac{1}{2}}+\frac{1}{16}n^{-\frac{1}{2}}-\frac{11}{48} \le \sum_{k=1}^{n}k^{\frac{1}{2}} \le \frac{2}{3}n^{\frac{3}{2}}+\frac{1}{2}n^{\frac{1}{2}}-\frac{1}{6}.
\end{aligned}$$

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Generating functions
2. Bernoulli polynomials and Bernoulli numbers
3. Properties of Bernoulli polynomials
4. Euler summation formula

Check your Progress:

1. $B_k{}'(t) =$ —————?

   (A) $kB_{k-1}(t),\ (k \ge 1)$ (B) $B_{k-1}(t),\ (k \ge 1)$

   (C) $(k-1)B_k(t),\ (k > 1)$ (D) None of these

2. Which of the following is not correct?

(A) $B_0 = 1$ (B) $B_2 = 1/6$ (C) $B_5 = 0$ (D) $B_{11} = -1/2$

3. For $m \geq 1, \quad B_{2m+1} = ...?$

(A) 1 (B) -1 (C) 0 (D) $-1/2$

Unit Summary:

In this unit, the definition and properties of difference operator and its inverse operator are provided. Also, Euler summation formula and its application are discussed. By studying these concepts, one can observe the differences and similarities between the difference and the differential calculus.

Glossary:

- Δ -The difference operator
- $\Delta^n y(t)$ -The nth order difference of $y(t)$
- E -The shift operator
- I -The identity operator
- $t^{\underline{r}}$ -The "falling factorial power" (read "t to the r falling")
- $\sum y(t)$ - An "indefinite sum" (or "antidifference") of $y(t)$
- $B_k(t)$ - Bernoulli polynomial
- B_k - Bernoulli number
- $\lfloor t \rfloor$ - "floor function" or the "greatest integer function"

Self-Assessment Questions:

1. Show that Δ and E commute-that is, $\Delta E y(t) = E \Delta y(t)$ for all $y(t)$.
2. Derive the formula

$$\Delta[x(t)y(t)z(t)] = \Delta x(t) E y(t) E z(t) + x(t) \Delta y(t) E z(t) + x(t) y(t) \Delta z(t)$$

Write down five other formulas of this type.

3.Show that

(a) $\Delta a^t = (a-1)a^t$ if a is a constant.

(b) $\Delta e^{ct} = (e^c - 1)\, e^{ct}$ if c is a constant.

4. Show that

(a) $\sum \cos at = \frac{\sin a\left(t-\frac{1}{2}\right)}{2\sin\frac{a}{2}} + C(t) \quad (a \neq 2n\pi)$.

(b) $\sum \binom{a+t}{t} = \binom{a+t}{t-1} + C(t)$, where $\Delta C(t) = 0$.

5. Let $z_n = \sum y_n$. Show that

$$\sum_{k=m}^{n-1} y_k = z_n - z_m.$$

6. Prove that $\int_0^1 B_k(t)dt = 0$ for $k \geq 1$.

Exercises:

1. Compute $\Delta\left(3^t \cos t\right)$ by two methods:

(a) Using Theorem 1.1.5.(d) and Theorem 1.1.6.(a) and (c).

(b) Directly from the definition of Δ.

2. Compute $\Delta^n t^{\underline{3}}$ and $\Delta^n t^3$ for $n = 1, 2, 3, \cdots$.

3. Find a solution of each of the following difference equations.

(a) $y(t+1) - y(t) = t^{\underline{3}} + 3^t$.

(b) $y(t+2) - 2y(t+1) + y(t) = \binom{t}{5}$.

4. Use summation by parts to compute $\sum t \sin t$.

5. Compute

$$\sum_{k=1}^{8} \frac{1}{(k+1)(k+2)(k+3)}$$

6. Give an estimate for $\sum_{k=1}^{400} k^{\frac{1}{2}}$.

Answers for Check your Progress:

Section 1.1	1. (B)	2. (B)	3. (A)
Section 1.2	1. (C)	2. (B)	3. (A)
Section 1.3	1. (A)	2. (D)	3. (C)

References:

1. W.G. Kelley and A.C. Peterson, “Difference Equations”, 2nd Edition, Academic Press, New York, 2001.

Suggested Reading:

1. R.P. Agarwal, “Difference Equations and Inequalities”, 2nd Edition, Marcel Dekker, New York, 2000.
2. S.N. Elaydi, “An Introduction to Difference Equations”, 3rd Edition, Springer, India, 2008.
3. R. E. Mickens, “Difference Equations”, 3rd Edition, CRC Press, 2015.

UNIT 2

Unit 2

Linear Difference Equations

Objectives:

This unit deals with the basic theory for linear difference equations and the method of solving them.

2.1 First Order Equations

Let $p(t)$ and $r(t)$ be given functions with $p(t) \neq 0$ for all t. The first order linear difference equation is

$$y(t+1) - p(t)y(t) = r(t). \tag{2.1}$$

Equation (2.1) is said to be of first order because it involves the values of y at t and $t+1$ only, as in the first order difference operator $\Delta y(t) = y(t+1) - y(t)$.

If $p(t) = 1$ for all t, then Eq. (2.1) is simply

$$\Delta y(t) = r(t),$$

and its solution is

$$y(t) = \sum r(t) + C(t),$$

where $\Delta C(t) = 0$.

Theorem 2.1.1. *Let $p(t) \neq 0$ and $r(t)$ be given for $t = a, a+1, \cdots$. Then*

(a) The solutions of the homogeneous equation

$$u(t+1) = p(t)u(t) \qquad for \quad t = a, a+1, \cdots \tag{2.2}$$

are

$$u(t) = u(a)\prod_{s=a}^{t-1} p(s), \quad (t = a+1, a+2, \cdots).$$

(b) All solutions of (2.1) are given by

$$y(t) = u(t)\left[\sum \frac{r(t)}{Eu(t)} + C\right],$$

where C is a constant and $u(t)$ is any nonzero function from part (a).

Proof. (a) From the "homogeneous" equation (2.2)

$$u(t+1) = p(t)u(t) \qquad for \quad t = a, a+1, \cdots,$$

we have

$$\begin{aligned} u(a+1) &= p(a)u(a) \\ u(a+2) &= p(a+1)p(a)u(a) \\ &\vdots \\ u(a+n) &= u(a)\prod_{k=0}^{n-1} p(a+k). \end{aligned}$$

Thus, we can write the solution as

$$u(t) = u(a)\prod_{s=a}^{t-1} p(s) \quad (t = a, a+1, \cdots),$$

where it is understood that $\prod_{s=a}^{a-1} p(s) \equiv 1$ and, for $t \geq a+1$, the product is taken over $a, a+1, \cdots, t-1$.

(b) Putting $y(t) = u(t)v(t)$ in equation (2.1), we get

$$\begin{aligned} u(t+1)v(t+1) - p(t)u(t)v(t) = r(t) \\ \implies u(t+1)v(t+1) - u(t+1)v(t) = r(t) \\ \implies \Delta v(t) = \frac{r(t)}{Eu(t)}. \end{aligned}$$

$$\implies v(t) \quad = \quad \sum \frac{r(t)}{Eu(t)} + C.$$

Thus, the solution of (2.1) is

$$y(t) = u(t)v(t) = u(t)\left[\sum \frac{r(t)}{Eu(t)} + C\right],$$

where C is an arbitrary constant and $u(t)$ is any nontrivial solution of equation (2.2).

□

Remark 2.1.2. *The method we used to solve (2.1) is a special case of the method of "variation of parameters".*

Example 2.1.3. *Find the solution $y(t)$ of*

$$y(t+1) - ty(t) = y(t+1)!, \quad (t = 1, 2, \cdots),$$

so that $y(1) = 5$.

By the previous theorem, the solution of the homogenous equation $u(t+1) - tu(t) = 0$ is given by

$$u(t) = u(1)\prod_{s=1}^{t-1} s = u(1)(t-1)!.$$

We can take u(1)=1.

Then, the solution of the non-homogenous equation is given by

$$\begin{aligned} y(t) &= u(t)\Big(\sum \frac{r(t)}{Eu(t)} + C\Big) \\ &= (t-1)!\left[\sum \frac{(t+1)!}{t!} + C\right] \\ &= (t-1)!\left[\sum (t+1) + C\right]. \end{aligned}$$

We know that $\sum t^k = \frac{1}{k+1}B_{k+1}(t) + C(t)$, where $\Delta C(t) = 0$.

Then

$$y(t) = (t-1)!\left[\frac{B_2(t+1)}{2} + C\right].$$

Using $B_2(t) = t^2 - t + \frac{1}{6}$, we have

$$\begin{aligned} y(t) &= (t-1)!\left[\frac{t(t+1)+\frac{1}{6}}{2}+C\right] \\ &= \frac{(t+1!)}{2}+\frac{(t-1!)}{12}+C(t-1)! \\ &= \frac{(t+1!)}{2}+D(t-1)!. \end{aligned}$$

To find D, take $t=1$.

Then

$$y(t)=5 \Rightarrow 1+D=5 \Rightarrow D=4.$$

Thus, the solution of the given non-homogeneous equation is

$$y(t) = \frac{(t+1)!}{2}+4(t-1)!, \quad (t=1,2,\cdots).$$

Example 2.1.4. *Suppose we deposit \$2000 at the beginning of each year in an IRA that pays an annual interest rate of 8%. How much will we have in the IRA at the end of the t^{th} year?*

Let $y(t)$ be the amount of money in the IRA at the end of the t^{th} year. Then

$$\begin{aligned} y(t+1) &= y(t)+(y(t)+2000)(0.08)+2000 \\ &= 1.08y(t)+2160. \end{aligned}$$

A solution of the homogeneous equation $u(t+1)=1.08u(t)$ is $u(t)=(1.08)^t$. Then

$$\begin{aligned} y(t) &= (1.08)^t\left[\sum \frac{2160}{(1.08)^{t+1}}+C\right] \\ &= (1.08)^t\left[\frac{2160}{1.08}\sum\left(\frac{1}{1.08}\right)^t+C\right]. \end{aligned}$$

We know that $\sum a^t=\frac{a^t}{a-1}$.

Then

$$\begin{aligned} y(t) &= (1.08)^t\left[\frac{2160}{1.08}\left(\frac{\left(\frac{1}{1.08}\right)^t}{\frac{1}{1.08}-1}\right)+C\right] \\ &= -27000+C(1.08)^t. \end{aligned}$$

Since $y(0)=0$, we have $C=27,000$, so that

$$y(t)=27000\big[(1.08)^t-1\big].$$

For example, at the end of twenty years, we would have

$$\begin{aligned} y(20) &= 27,000\left[(1.08)^{20} - 1\right] \\ &\approx \$98,845.84. \end{aligned}$$

Example 2.1.5. *Find the solution $y(t)$ of*

$$y(t+1) - ty(t) = 1, \quad (t = 1, 2, \cdots),$$

so that $y(1) = 1 - e$.

First, note that the solutions of $u(t+1) - tu(t) = 0$ are

$$u(t) = u(1) \prod_{s=1}^{t-1} s = u(1)(t-1)!.$$

We can take $u(t) = 1$. Then

$$y(t) = (t-1)! \left[\sum \frac{1}{t!} + C\right].$$

We know that $\sum y_n = \sum_{k=m}^{n-1} y_k + C$. Then

$$y(t) = (t-1)! \left[\sum_{k=1}^{t-1} \frac{1}{k!} + C\right].$$

To evaluate C, let $t = 1$. Then, we get

$$\begin{aligned} y(1) &= (1-1)! \left[\sum_{k=1}^{1-1} \frac{1}{k!} + C\right] \\ \Rightarrow C &= 1 - e. \end{aligned}$$

Therefore, the exact solution is given by

$$y(t) = (t-1)! \left[1 - e + \sum_{k=1}^{t-1} \frac{1}{k!}\right].$$

Alternate Method I

Now, consider

$$y(t+1) - p(t)y(t) = r(t) \tag{2.3}$$

and

$$u(t+1) = p(t)u(t). \tag{2.4}$$

Let us solve the above equation (2.3) and (2.4) for t in a discrete or continuous domain.

For simplicity, we assume $p(t) > 0$.

Applying the natural logarithm to both sides of equation (2.4), we get

$$\begin{aligned} \log|u(t+1)| &= \log|u(t)| + \log p(t), \\ \Longrightarrow \Delta \log|u(t)| &= \log p(t), \\ \Longrightarrow \log|u(t)| &= \sum \log p(t) + D(t), \end{aligned}$$

where $\Delta D(t) = 0$.

Then

$$\begin{aligned} |u(t)| &= e^{D(t)} e^{\sum \log p(t)}, \\ \Longrightarrow u(t) &= C(t) e^{\sum \log p(t)}, \end{aligned}$$

where $\Delta C(t) = 0$.

Once $u(t)$ is found, the solution $y(t)$ of equation (2.3) can be computed using Theorem 2.1.1(b) with the constant C replaced by an arbitrary function $C(t)$ so that $\Delta C(t) = 0$.

Example 2.1.6. *Solve the equation*

$$u(t+1) = a\frac{(t-r_1)(t-r_2)\cdots(t-r_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}u(t),$$

where $a, r_1, r_2, \cdots, r_n, s_1, s_2, \cdots, s_m$ are constants.

Assume that all factors in the preceding expression are positive. Then

$$u(t+1) = a\frac{(t-r_1)(t-r_2)\cdots(t-r_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}u(t).$$

Taking $\log$ on both sides, we get

$$\begin{aligned} \log(u(t+1)) &= \log\left[a\frac{(t-r_1)(t-r_2)\cdots(t-r_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}u(t)\right] \\ \Longrightarrow \log(u(t+1)) &= \log\left[a\frac{(t-r_1)(t-r_2)\cdots(t-r_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}\right] + \log\ u(t) \end{aligned}$$

$$\Longrightarrow \log(u(t+1)) - \log u(t) = \log\left[a\frac{(t-r_1)(t-r_2)\cdots(t-r_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}\right]$$

$$\Longrightarrow \Delta \log u(t) = \log\left[a\frac{(t-r_1)(t-r_2)\cdots(t-r_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}\right].$$

The solution is

$$\log u(t) = \sum \log\left[a\frac{(t-r_1)(t-r_2)\cdots(t-r_n)}{(t-s_1)(t-s_2)\cdots(t-s_m)}\right] + D(t),$$

where $D(t) = 0$*.*

Then

$$\begin{aligned}\log u(t) &= \sum\Big[\log a + \log(t-r_1) + \cdots + \log(t-r_n) - \log(t-s_1) - \cdots \\ &\quad - \log(t-s_m)\Big] + D(t)\end{aligned}$$

$$\Longrightarrow \quad u(t) = e^{D(t)} e^{\sum\left[\log a + \log(t-r_1)+\cdots+\log(t-r_n)-\log(t-s_1)-\cdots-\log(t-s_m)\right]}.$$

We know that $\sum \log t = \log\Gamma(t) + C(t)$*.*

Then

$$u(t) = C(t)e^{\left[t\log a + \log\Gamma(t-r_1)+\cdots+\log\Gamma(t-r_n)-\Gamma\log(t-s_1)-\cdots-\Gamma\log(t-s_m)\right]}$$

$$\Longrightarrow u(t) = C(t)a^t\frac{\Gamma(t-r_1)\cdots\Gamma(t-r_n)}{\Gamma(t-s_1)\cdots\Gamma(t-s_m)},$$

where $\Delta C(t) = 0$*.*

By direct substitution, we can show that this expression for $u(t) = 0$ *solves the difference equation for all values of* t *where the various gamma functions are defined. We can conclude that equation (2.2) is solvable in terms of gamma functions if* $p(t)$ *is a rational function.*

Example 2.1.7. *Consider*

$$u(t+1) = \frac{t}{2t^2+3t+1}\,u(t).$$

The coefficient function factors as follows:

$$\frac{t}{2t^2+3t+1}\,u(t) = \frac{1}{2}\frac{t}{(t+1)(t+\frac{1}{2})}.$$

$\therefore$ *By Example 2.1.6, we have*

$$\begin{aligned} u(t) &= C(t)\Big(\frac{1}{2}\Big)^t \frac{\Gamma(t)}{\Gamma(t+1)\Gamma(t+\frac{1}{2})} \\ &= C(t)\Big(\frac{1}{2}\Big)^t \frac{1}{t\Gamma(t+\frac{1}{2})}. \end{aligned}$$

Alternate Method II

Let's rewrite equation (2.1) in the fractional form

$$y(t) = \frac{-r(t) + y(t+1)}{p(t)}. \tag{2.5}$$

Then

$$y(t+1) = \frac{-r(t+1) + y(t+2)}{p(t+1)}. \tag{2.6}$$

Substituting (2.6) in (2.5), we get

$$y(t) = \frac{-r(t) + \frac{-r(t+1)+y(t+2)}{p(t+1)}}{p(t)}.$$

Continuing in this way, we obtain the continued fraction

$$y(t) = \frac{-r(t) + \frac{-r(t+1)+\frac{-r(t+2)+\frac{-r(t+3)+\cdots}{p(t+3)}}{p(t+2)}}{p(t+1)}}{p(t)}.$$

If we formally divide out the continued fraction, we arrive at the infinite series

$$y(t) = \frac{-r(t)}{p(t)} + \frac{-r(t+1)}{p(t)p(t+1)} + \cdots,$$

or

$$y(t) = \sum_{k=0}^{\infty} \frac{-r(t+k)}{p(t)\cdots p(t+k)}. \tag{2.7}$$

When this series converges, its sum must be a solution of equation (2.1).

Example 2.1.8. *Consider the equation*

$$y(t+1) - ty(t) = -3^t.$$

By equation (2.7), we have

$$y(t) = \sum_{k=0}^{\infty} \frac{3^{t+k}}{t(t+1)\cdots(t+k)} = \frac{3^t}{t}\sum_{k=0}^{\infty} 3^t t^{\underline{-k}},$$

a "factorial series." The ratio test shows that this converges for all $t \neq 0, -1, -2, \cdots$, so the series represents one solution of the difference equation.

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Solution of first order homogeneous equations
2. Solution of first order nonhomogeneous equations

Check your Progress:

1. A solution of the first order linear difference equation $\Delta y(t) = r(t)$ is ....?

   (A) $y(t) = r(t)$ (B) $y(t) = \sum r(t)$ (C) $\sum y(t) = r(t)$ (D) None of these

2. $u(t) = u(a)\prod_{s=a}^{t-1} p(s)$ is a solution of the equation ....

   (A) $u(t) = p(t)u(t+1)$ (B) $u(t+1) = p(t)u(t)$

   (C) $u(t+a) = p(t)$ (D) None of these

3. If $\Delta \log |u(t)| = \log p(t)$, then

   (A) $\log |u(t)| = \log p(t)$ (B) $\log u(t+1) = \sum \log p(t)$

   (C) $\log |u(t)| = \sum \log p(t)$ (D) None of these

2.2 General Results for Linear Equations

The linear equation of the n^{th} order is

$$p_n(t)y(t+n) + \cdots + p_0(t)y(t) = r(t), \tag{2.8}$$

where $p_0(t), \cdots, p_n(t)$ and $r(t)$ are assumed to be known and $p_0(t) \neq 0, p_n(t) \neq 0$ for all t. If $r(t) \not\equiv 0$, we say that (2.8) is "nonhomogeneous." As in last section, we will study (2.8) in association with the corresponding homogeneous equation

$$p_n(t)u(t+n) + \cdots + p_0(t)u(t) = 0. \tag{2.9}$$

Note that (2.8) can also be written using the shift operator as

$$\left(p_n(t)E^n + \cdots + p_0(t)E^0\right) y(t) = r(t),$$

where $E^0 = I$. Since $E = \Delta + I$, it is also possible to write (2.8) in terms of the difference operator. However, the following example shows that the order of the equation is not apparent in that case.

Example 2.2.1. *What is the order of the equation*

$$\Delta^3 y(t) + 3\Delta^2 y(t) + \Delta y(t) - y(t) = r(t)?$$

Put $\Delta = E - I$ and expand the power of Δ:

$$\Longrightarrow (E-I)^3 y(t) + 3(E-I)^2 y(t) + (E-I)y(t) - y(t) = r(t)$$

$$\Rightarrow \left(E^3 - 3E^2 + 3E - I\right) y(t) + 3\left(E^2 - 2E + I\right) y(t) \\ + (E - I)\, y(t) - y(t) = r(t)$$

$$\Rightarrow E^3 y(t) - 2Ey(t) = r(t)$$

or

$$y(t+3) - 2y(t+1) = r(t).$$

$\therefore$ *The order of the given difference equation is*

$$(t+3) - (t+1) = 2.$$

Theorem 2.2.2. *Assume that $p_0(t), \cdots, p_n(t)$, and $r(t)$ are defined for $t = a, a+1, \cdots$ and $p_0(t) \neq 0, p_n(t) \neq 0$ for all t. Then for any t_0 in $\{a, a+1, \cdots\}$ and any numbers $y_0, \cdots, y_{n-1}$, there is exactly one $y(t)$ that satisfies (2.8) for $t = a, a+1, \cdots$ and $y(t_0 + k) = y_k$ for $k = 0, \cdots, n-1$.*

Proof. The proof follows from iteration. For example,

$$y(t_0 + n) = \frac{r(t_0) - p_{n-1}(t_0)\, y_{n-1} - \cdots - p_0(t_0)\, y_0}{p_n(t_0)}$$

since $p_n(t_0) \neq 0$. Similarly, we can solve (2.8) for $y(t)$ when $t > t_0 + n$ in terms of the n preceding values of y. Since $p_0(t)$ is never 0 , we can also solve for $y(t)$ when $t < t_0$. □

Theorem 2.2.3. *(a) If $u_1(t)$ and $u_2(t)$ solve (2.9) , then so does $Cu_1(t) + Du_2(t)$ for any constants C and D.*

(b) If $u(t)$ solves (2.9) and $y(t)$ solves (2.8) , then $u(t) + y(t)$ solves (2.8) .

(c) If $y_1(t)$ and $y_2(t)$ solve (2.8) , then $y_1(t) - y_2(t)$ solves (2.9) .

Proof. (a) By our assumption, we have

$$p_n(t)u_1(t+n) + \cdots + p_0(t)u_1(t) = 0 \tag{2.10}$$

and

$$p_n(t)u_2(t+n) + p_{n-1}u_2(t+n-1) + \cdots + p_0(t)u_2(t) = 0. \tag{2.11}$$

Multiply equation (2.10) by the constant C to obtain

$$C\left(p_n(t)u_1(t+n) + \cdots + p_0(t)u_1(t)\right) = 0.$$

Multiply equation (2.11) by the constant D to obtain

$$D\left(p_n(t)u_2(t+n) + \cdots + p_0(t)u_2(t)\right) = 0.$$

Adding the above two equations, we have

$$\begin{aligned} Cp_n(t)u_1(t+n) + \cdots + Cp_0(t)u_1(t) + Dp_n(t)u_2(t+n) + \\ \cdots + Dp_0(t)u_2(t) &= 0 \\ \Longrightarrow p_n(t)\left(Cu_1(t+n) + Du_2(t+n)\right) + \cdots + p_0(t)\left(Cu_1(t) + Du_2(t)\right) &= 0. \end{aligned}$$

This can be rewritten as

$$p_n(t)u(t+n) + \cdots + p_0(t)u(t) = 0, \tag{2.12}$$

where $u(t) = Cu_1(t) + Du_2(t)$.

That is, $Cu_1(t) + Du_2(t)$ satisfies equation (2.12)

$\therefore Cu_1(t) + Du_2(t)$ is a solution of equation (2.12).

(b) By our assumption, we have

$$p_n(t)u(t+n) + \cdots + p_0(t)u(t) = 0$$

and

$$p_n(t)y(t+n) + \cdots + p_0(t)y(t) = r(t).$$

Adding the above two equations, we get

$$\begin{aligned} p_n(t)u(t+n) + \cdots + p_0(t)u(t) + p_n(t)y(t+n) + \cdots + p_0(t)y(t) &= r(t) \\ \Longrightarrow p_n(t)\left(u(t+n) + y(t+n)\right) + \cdots + p_0(t)\left(u(t) + y(t)\right) &= r(t). \end{aligned}$$

This can be rewritten as

$$p_n(t)Y(t+n) + \cdots + p_0(t)Y(t) = r(t), \tag{2.13}$$

where $Y(t) = u(t)y(t)$.

That is, $u(t)y(t)$ satisfies equation (2.13).

$\therefore u(t)y(t)$ is a solution of equation (2.13).

(c) By our assumption, we have

$$p_n(t)y_1(t+n) + \cdots + p_0(t)y_1(t) = r(t)$$

and

$$p_n(t)y_2(t+n) + \cdots + p_0(t)y_2(t) = r(t).$$

Subtracting the above two equations, we get

$$p_n(t)\left(y_1(t+n) - y_2(t+n)\right) + \cdots + p_0(t)\left(y_1(t) - y_2(t)\right) = 0.$$

This can be rewritten as

$$p_n(t)u(t+n) + \cdots + p_0(t)u(t) = 0, \tag{2.14}$$

where $u(t) = y_1(t) - y_2(t)$.

That is $y_1(t) - y_2(t)$ satisfies equation (2.14).

$\therefore y_1(t) - y_2(t)$ is a solution of equation (2.14).

□

Corollary 2.2.4. *If $z(t)$ is a solution of (2.8), then every solution $y(t)$ of (2.8) takes the form*

$$y(t) = z(t) + u(t),$$

where $u(t)$ is some solution of Eq. (2.9).

Proof. This is just a restatement of Theorem 2.2.3 (c). □

Remark 2.2.5. *As a result of Corollary (2.2.4) the problem of finding all solutions of Eq. (2.8) reduces to two smaller problems:*

(a) Find all solutions of Eq. (2.9).

(b) Find one solution of Eq. (2.8).

This simplification is identical to that for linear differential equations. To analyze the first problem, we need some definitions.

Definition 2.2.6. *The set of functions $\{u_1(t), \cdots, u_m(t)\}$ is "linearly dependent" on the set $t = a, a+1, \cdots$ if there are constants $C_1, \cdots, C_m$, not all zero, so that*

$$C_1u_1(t) + C_2u_2(t) + \cdots + C_mu_m(t) = 0$$

for $t = a, a+1, \cdots$. Otherwise, the set is said to be "linearly independent."

Example 2.2.7. *Show that the functions $2^t, t2^t$ and t^22^t are linearly independent on every set $t = a, a+1, \ldots,$.*

Suppose that there are constants $\mathrm{C}_1, \mathrm{C}_2, \mathrm{C}_3$ such that

$$\begin{aligned} C_12t + C_2t2^t + C_3t^22 &= 0, \quad t = a, a+1, \cdots \\ \Rightarrow C_1 + C_2t + C_3t^2 &= 0, \quad t = a, a+1, \cdots \end{aligned}$$

This is possible only if $C_1 = C_2 = C_3 = 0$.

Therefore, $2^t, t2^t, t^22^t$are linearly Independent on $\{a, a+1, \ldots\}$.

Example 2.2.8. *Show that the functions $u_1(t) = 2, u_2(t) = 1 + \cos \pi t$ are linearly independent on the set $t = 1, 2, 3, \ldots$.*

Suppose that

$$\begin{aligned} C_1u_1(t) + C_2u_1(t) &= 0 \\ \Longrightarrow 2C_1 + C_2(1 + \cos \pi t) &= 0 \\ \Longrightarrow 2C_1 + C_2 + C_2 \cos \pi t &= 0, \quad t = 1, 2, 3, \ldots. \end{aligned}$$

When $t=1, \quad 2C_1=0, \Rightarrow \quad C_1=0.$

When $t=2, \quad 2C_1+2C_2=0, \Rightarrow C_2=0.$

$\therefore u_1$ *and* u_2 *are linearly independent on*$\{1,2,3,\ldots\}$.

Example 2.2.9. *Suppose that* $u_1(t)=2, u_2(t)=1+\cos\pi t$ *are linearly dependent on* $\{\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots\}$.

Note that

$$\begin{aligned} u_1(t)-2u_2(t) &= 2-2(1+\cos\pi t) \\ &= 0 \end{aligned}$$

for $t=\frac{1}{2},\frac{3}{2},\frac{5}{2},\cdots$.

$\Longrightarrow \; u_1(t)=2u_2(t) \quad$ *for* $\quad t=\frac{1}{2},\frac{3}{2}\ldots$.

$\therefore u_1$ *and* u_2*-are linearly dependent on* $\left\{\frac{1}{2},\frac{3}{2},\ldots.\right\}$.

Definition 2.2.10. *The matrix of Casorati is given by*

$$W(t)=\begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_n(t) \\ u_1(t+1) & u_2(t+1) & \cdots & u_n(t+1) \\ \vdots & \vdots & \ddots & \vdots \\ u_1(t+n-1) & \cdots & \cdots & u_n(t+n-1) \end{bmatrix},$$

where $u_1,\cdots,u_n$ *are given functions. The determinant*

$$w(t)=\det W(t)$$

is called the "Casoratian."

It is easy to check that the Casoratian satisfies the equation

$$w(t)=\det\begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_n(t) \\ \Delta u_1(t) & \Delta u_2(t) & \cdots & \Delta u_n(t) \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{n-1}u_1(t) & \cdots & \cdots & \Delta^{n-1}u_n(t) \end{bmatrix}. \tag{2.15}$$

Theorem 2.2.11. *Let* $u_1(t),\cdots,u_n(t)$ *be solutions of* (2.9) *for* $t=a,a+1,\cdots$. *Then the following statements are equivalent:*

(a) The set $\{u_1(t),\cdots,u_n(t)\}$ *is linearly dependent for* $t=a,a+1,\cdots$.

(b) $w(t)=0$ *for some* t.

(c) $w(t)=0$ *for all* t.

Proof. First suppose that $u_1(t), u_2(t), \cdots, u_n(t)$ are linearly dependent. Then there are constants $C_1, C_2, \cdots, C_n$, not all zero, so that

$$\begin{aligned} C_1u_1(t) + C_2u_2(t) + \cdots + C_nu_n(t) &= 0, \\ C_1u_1(t+1) + C_2u_2(t+1) + \cdots + C_nu_n(t+1) &= 0, \\ &\vdots \\ C_1u_1(t+n-1) + C_2u_2(t+n-1) + \cdots + C_nu_n(t+n-1) &= 0, \end{aligned}$$

for $t = a, a+1, \cdots$.

Since this homogeneous system has a nontrivial solution $C_1, C_2, \cdots, C_n$, the determinant of the matrix of coefficients $w(t)$ is zero for $t = a, a+1, \cdots$.

Conversely, suppose that $w(t_0) = 0$. Then there are constants $C_1, C_2, \cdots, C_n$, not all zero, so that

$$\begin{aligned} C_1u_1(t_0) + C_2u_2(t_0) + \cdots + C_nu_n(t_0) &= 0, \\ C_1u_1(t_0+1) + C_2u_2(t_0+1) + \cdots + C_nu_n(t_0+1) &= 0, \\ &\vdots \\ C_1u_1(t_0+n-1) + C_2u_2(t_0+n-1) + \cdots + C_nu_n(t_0+n-1) &= 0. \end{aligned}$$

Let

$$u(t) = C_1u_1(t) + C_2u_2(t) + \cdots + C_nu_n(t).$$

Then u is a solution of Eq. (2.9) and

$$u(t_0) = u(t_0+1) = \cdots = u(t_0+n-1) = 0.$$

It follows immediately from Theorem 2.2.2 that $u(t) = 0$ for all t, hence the set $\{u_1, u_2, \cdots, u_n\}$ is linearly dependent. □

The importance of the linear independence of solutions to (2.9) is a consequence of the next theorem.

Theorem 2.2.12. *If $u_1(t), \cdots, u_n(t)$ are independent solutions of (2.9), then every solution $u(t)$ of (2.9) can be written in the form*

$$u(t) = C_1u_1(t) + \cdots + C_nu_n(t)$$

for some constants $C_1, \cdots, C_n$.

Proof. Let $u(t)$ be a solution of (2.9). Since $w(t) \neq 0$ for $t = a, a+1, \cdots$, the system of equations

$$C_1 u_1(a) + \cdots + C_n u_n(a) = u(a),$$
$$\vdots$$
$$C_1 u_1(a+n-1) + \cdots + C_n u_n(a+n-1) = u(a+n-1)$$

has a unique solution $C_1, \cdots, C_n$. Recall that a solution of (2.9) is uniquely determined by its values at $t = a, a+1, \cdots, a+n-1$, so we must have

$$u(t) = C_1 u_1(t) + \cdots + C_n u_n(t),$$

for all t. □

Example 2.2.13. *The equation*

$$u(t+3) - 6u(t+2) + 11u(t+1) - 6u(t) = 0$$

has solutions $2^t, 3^t, 1$ *for all values of* t*. Their Casoratian is from Eq.* (2.15)

$$w(t) = \det \begin{bmatrix} 2^t & 3^t & 1 \\ 2^t & 2 \cdot 3^t & 0 \\ 2^t & 4 \cdot 3^t & 0 \end{bmatrix} = 2^{t+1} 3^t,$$

which does not vanish. Consequently, the set $\{2^t, 3^t, 1\}$ *is linearly independent, and all solutions of the equation have the form*

$$u(t) = C_1 2^t + C_2 3^t + C_3.$$

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. n^{th} order initial value problem
2. Linearly independent solutions
3. Matrix of Casorati
4. Role of Casoratian

Check your Progress:

1. What is the order of the equation $\Delta^3 y(t) + \Delta^2 y(t) - \Delta y(t) - y(t) = 0$?

   (A) 0 (B) 1 (C) 2 (D) 3

2. The operator form of the difference equation $y(t+2) - 7y(t+1) + 6y(t) = t$ is

   (A) $(E-1)(E-6)y(t) = t$ (B) $(E+1)(E+6)y(t) = t$

   (C) $(E+1)(E-6)y(t) = t$ (D) $(E-1)(E+6)y(t) = t$

3. The Casoratian of the functions $2^t, 3^t, 1$ is

   (A) $2^t 3^{t+1}$ (B) $2^t 3^t$ (C) $2^{t^2} 3^t$ (D) $2^{t+1} 3^t$

2.3 Solving Linear Equations

In this section, we are going to find the n linearly independent solutions of the homogeneous equations $p_n u(t+n) + p_{n-1} u(t+n-1) + \cdots + p_0 u(t) = 0$, where $p_0, p_1, \ldots, p_{n-1}$ are constants.

Since $p_n \neq 0$, we can divide the above equation by p_n are relabel the resulting equation to obtain.

$$u(t+n) + p_{n-1} u(t+n-1) + \cdots + p_0 u(t) = 0, \tag{2.16}$$

where $p_0, p_1, \ldots, p_{n-1}$ are constants and $p_0 \neq 0$.

Definition 2.3.1. *(a) The polynomial $\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0$ is called the "characteristic polynomial" for (2.16).*

(b) The equation $\lambda^n + \cdots + p_0 = 0$ is the "characteristic equation" for (2.16).

(c) The solutions $\lambda_1, \cdots, \lambda_k$ of the characteristic equation are the "characteristic roots."

Theorem 2.3.2. *Suppose that (2.16) has characteristic roots $\lambda_1, \cdots, \lambda_k$ with multiplicities $\alpha_1, \cdots, \alpha_k$, respectively. Then (2.16) has the n independent solutions*

$$\lambda_1^t, \cdots, t^{\alpha_1 - 1}\lambda_1^t, \lambda_2^t, \cdots, t^{\alpha_2 - 1}\lambda_2^t, \cdots, \lambda_k^t, \cdots, t^{\alpha_k - 1}\lambda_k^t.$$

Proof. Consider $u(t+n) + p_{n-1}u(t+n-1) + \cdots + p_0 u(t) = 0$.

Using shift operator E, it can be written as

$$\begin{aligned} \left(E^n + p_{n-1}E^{n-1} + \cdots + p_0\right) u(t) &= 0. \\ (i.e)\,(E-\lambda_1)^{\alpha_1}\,(E-\lambda_2)^{\alpha_2} \ldots (E-\lambda_k)^{\alpha_k}\, u(t) &= 0. \end{aligned} \tag{2.17}$$

Since $p_0 \neq 0$, each characteristic root is non-zero.

Let us solve the equation

$$(E-\lambda_1)^{\alpha_1}\, u(t) = 0. \tag{2.18}$$

$\therefore$ If $\alpha_1 = 1$, then (2.18) becomes

$$\begin{aligned} (E-\lambda_1)\, u(t) &= 0. \\ (i.e) u(t+1) - \lambda_1 u(t) &= 0. \end{aligned}$$

It's solution is $u(t) = \lambda_1^t$, if $u(1) = \lambda_1$.

If $\alpha_1 > 1$, let $u(t) = \lambda_1^t v(t)$ in (2.18).

$$\begin{aligned} \text{Then } (E-\lambda_1)^{\alpha_1}\,\lambda^t v(t) &= \sum_{i=0}^{\alpha_1} \binom{\alpha_1}{i} (-\lambda_1)^{\alpha_1 - i}\, E^i \lambda_1^t v(t) \\ &= \sum_{i=0}^{\alpha_1} \binom{\alpha_1}{i} \cdot (-\lambda_1)^{\alpha_1 - i}\, \lambda_1^{t+i} \quad E^i v(t) \\ &= x_1^{\alpha_1 + t} \sum_{i=0}^{\alpha_1} \binom{\alpha_1}{i} (-1)^{\alpha_1 - i} E^i v(t) \\ &= \lambda_1^{\alpha_1 + t} \cdot (E-1)^{\alpha_1} v(t) \\ &= \lambda_1^{\alpha_1 + t} \Delta^{\alpha_1} v(t) \\ &= 0 \quad \text{if} \quad v(t) = 1, t, t^2, \ldots, t^{\alpha_1 - 1}. \end{aligned}$$

Thus (2.16) has α_1 solutions of the form $\lambda_1^t, t\lambda_1^t, t^2\lambda_1^t, \ldots, t^{\alpha_1 - 1}\lambda_1^t$. □

Example 2.3.3. *Find all the solutions of*

$$u(t+3) - 7u(t+2) + 16u(t+1) - 12u(t) = 0, t = a, a+1, \ldots. \tag{2.19}$$

The characterise equation is

$$\lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0.$$

$$(\lambda - 2)^2(\lambda - 3) = 0.$$

$\therefore$ *The three independent solutions of* (2.19) *are*

$$u_1(t) = 2^t, \quad u_2(t) = t2^t, u_3(t) = 3^t.$$

Let's verify independence:

$$\begin{aligned}
w(t) &= \begin{vmatrix} 2^t & t2^t & 3^t \\ 2^{t+1} & (t+1)2^{t+1} & 3^{t+1} \\ 2t+2 & (t+2)2^{t+2} & 3^{t+2} \end{vmatrix} \\
&= 2^t 2^t 3^t \begin{vmatrix} 1 & t & 1 \\ 2 & 2(t+1) & 3 \\ 4 & 4(t+2) & 9 \end{vmatrix} \\
&= 2^{2t}3^t\{(18t+18-12t-24)-t(18-12)+(8t+16-8t-8) \\
&= 2^{2t}3^t \left\{b^E - b - 6b + 8\right\} \\
&= 2^{2t+1}3^t
\end{aligned}$$

$$w(t) \neq 0.$$

$\therefore$ *The general solutions of the difference equation is*

$$u(t) = C_1 2^t + C_2 t 2^t + C_3 3^t, \text{ where } C_1, C_2 \text{ and } C_3$$

are arbitrary constants.

Example 2.3.4. *Find independent real solutions of*

$$u(t+2) - 2u(t+1) + 4u(t) = 0. \tag{2.20}$$

The characteristic equation is $\lambda^2 - 2\lambda + 4 = 0.$

$$\begin{aligned}
&\Longrightarrow \lambda = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm i\sqrt{12}}{2} \\
&\Longrightarrow \lambda = 1 \pm i\sqrt{3} \\
&\Longrightarrow \lambda = re^{\pm i\theta} = r(\cos\theta \pm i\sin\theta) = 2\left(\cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3}\right) \\
&\quad \therefore \lambda^t = 2^t\left(\cos\frac{\pi}{3}t \pm i\sin\frac{\pi}{3}t\right).
\end{aligned}$$

$\therefore$ *The two real solutions are*

$$u_1(t) = 2^t\cos\frac{\pi}{3}t, \quad u_2(t) = 2^t\sin\frac{\pi}{3}t.$$

$$\begin{aligned}
\textit{Now, } \omega(t) &= \begin{vmatrix} 2^t \cos \frac{\pi}{3} t & 2^t \sin \frac{\pi}{3} t \\ 2^{t+1} \cos \frac{\pi}{3}(t+1) & 2^{(t)} \sin \frac{\pi}{3}(t+1) \end{vmatrix} \\
&= 2^{2t+1} \left[\sin \frac{\pi}{3}(t+r) \cos \frac{\pi}{3} t - \cos \frac{\pi}{3}(t+1) \sin \frac{\pi}{3} t \right] \\
&= 2^{2t+1} \sin \left[\frac{\pi}{3}(t+1) - \frac{\pi}{3} t \right] \\
&= 2^{2t+1} \frac{\sqrt{3}}{2} \neq 0.
\end{aligned}$$

$\therefore$ *u_1 and u_2 are the independent real solutions of* (2.20).

Annihilator Method:

The general non-homogeneous equation with constant coefficients.

$$y(t+n) + p_{n-1} y(t+n-1) + \ldots + p_0 y(t) = r(t)$$

can be solved by "annihilator method" if $r(t)$ is a solution of some homogenous equation with constant coefficient.

Theorem 2.3.5. ***(Annihilator Method)*** *Suppose that y (t) solves,*

$$y(t+n) + p_{n-1} y(t+n-1) + \ldots + p_0 y(t) = r(t) \tag{2.21}$$

and that $r(t)$ satisfies

$$\left(E^m + q_{m-1} E^{m-1} + \cdots + q_0\right) r(t) = 0.$$

Then $y(t)$ satisfies

$$\left(E^m + q_{m-1} E^{m-1} + \cdots + q_0\right) \left(E^n + \ldots + p_0\right) y(t) = 0.$$

(Here, $E^m + q_{m-1} E^{m-1} + \cdots + q_0$ is called the annihilator.)

Proof. Using the shift operator E, (2.21) can be written as

$$\left(E^n + p_{n-1} E^{n-1} + \cdots \cdot + p_a\right) y(t) = r(t).$$

Applying $(E^m + q_{m-1} E^{m-1} + \cdots + q_0)$ on both sides, we get

$$\begin{aligned}
\left(E^m + q_{m-1} E^{m-1} + \cdots + q_0\right) \left(E^n + p_{n-1} E^{n-1} + \cdots + p_0\right) y(t) &= \left(E^m + q_{m-1} E^{m-1} + \ldots + q_0\right) r(t) \\
&= 0.
\end{aligned}$$

Hence, the theorem. $\square$

Example 2.3.6. *Solve* $y(t+2) - 7y(t+1) + 6y(t) = t$.

The given difference equation can be written as

$$\begin{aligned} \left(E^2 - 7E + 6\right) y(t) &= t. \\ \implies (E-1)(E-6)y(t) &= t. \end{aligned}$$

Now, t *satisfies the homogeneous equation*

$$(E-1)^2 t = \Delta^2 t = 0.$$

Here, $E-1)^2$ *is the annihilator.*

Then, by the previous theorem, $y(t)$ *satisfies*

$$(E-1)^2(E-1)(E-6)y(t) = 0.$$

$$\implies (E-1)^3(E-6)y(t) = 0.$$

$$\therefore y(t) = C_1 6^t + C_2 + C_3 t + C_4 t^2.$$

Substitute the value of $y(t)$ *in the given difference equation. Then*

$$\begin{aligned} C_1 6^{t+2} + C_2 + c_3(t+2) + C_4(t+2)^2 - 7C_1 6^{t+1} - 7C_2 & \\ -7C_3(t+1) - 7C_4(t+1)^2 + 6C_1 6^t + 6C_2 + 6C_3 t + 6C_4 t^2 &= t \\ \Rightarrow (6^{t+2} - 76^{t+1} + 6^{t+1}) - 5C_3 + (-10t-3)C_4 &= t \\ \Rightarrow 6^{t+1} C_1[6 - 7 + 1] - 5C_3 + (-10t-3)C_4 &= t \\ \Rightarrow -10C_4 t + (-5C_3 - 3C_4) &= t. \end{aligned}$$

Equating the coefficients of t*, we get*

$$\begin{aligned} -10C_4 &= 1 \\ \Rightarrow C_4 &= \frac{-1}{10}. \end{aligned}$$

Also, equating the constants, we get

$$\begin{aligned} -5C_3 - 3C_4 &= 0 \\ \Rightarrow 5C_3 &= \frac{-3}{10} \\ \Rightarrow C_3 &= \frac{3}{50}. \end{aligned}$$

Thus,

$$y(t) = C_1 6^t + C_2 + \frac{3}{50}t - \frac{1}{10}t^2.$$

Example 2.3.7.

$$\text{Solve } \Delta y(t) = 3^t \sin\frac{\pi}{2}t, (t = a, a+1, \cdots). \tag{2.22}$$

The given difference equation can be written as

$$(E-1)y(t) = 3^t \sin\frac{\pi}{2}t.$$

Now, $3^t \sin\frac{\pi}{2}t$ *must satisfy an equation with complex roots.*

The polar coordinates of $3^t \sin\frac{\pi}{2}t, 3^t \cos\frac{\pi}{2}t$ *are* $r = 3$ *and* $\theta = \pm\frac{\pi}{2}$.

$$\begin{aligned} \therefore \lambda &= 3^{\pm\frac{\pi}{2}i} \\ &= \pm 3i. \end{aligned}$$

$\implies$ $3^t \sin\frac{\pi}{2}t$ *satisfies the homogeneous equation*

$$\begin{aligned} (E-3i)(E+3i)3^t \sin\frac{\pi}{2}t &= 0. \\ \implies \qquad \left(E^2+9\right) 3^t \sin\frac{\pi}{2}t &= 0. \end{aligned}$$

Thus $y(t)$ *satisfies*

$$\left(E^2+9\right)(E-1)y(t) = 0.$$

$\therefore$ *The general solution is given by*

$$y(t) = C_1 + C_2 3^t \sin\frac{\pi}{2}t + C_3 3^t \cos\frac{\pi}{2}t.$$

Substituting this expression for $y(t)$ *in equation* (2.22), *we get*

$$\begin{aligned} C_1 + C_2 3^{t+1} \sin\frac{\pi}{2}(t+1) + C_3 3^{t+1} \cos\frac{\pi}{2}(t+1) - C_1 - C_2 3^t \sin\frac{\pi}{2}t - C_3 3^t \cos\frac{\pi}{2}t &= 3^t \sin\frac{\pi}{2}t \\ \Rightarrow C_2 3^t \left(3\sin\frac{\pi}{2}t\cos\frac{\pi}{2} + 3\cos\frac{\pi}{2}t\sin\frac{\pi}{2} - \sin\frac{\pi}{2}t\right) & \\ + C_3 3^t \left(3\cos\frac{\pi}{2}t\cos\frac{\pi}{2} - 3\sin\frac{\pi}{2} + \sin\frac{\pi}{2} - \cos\frac{\pi}{2}t\right) &= 3^t \sin\pi/2t \\ \Rightarrow C_2 3^t \left(3\cos\frac{\pi}{2}t - \sin\frac{\pi}{2}t\right) + C_3 3^t \left(-3\sin\frac{\pi}{2}t - \cos\pi\right. &= 3^t \sin\frac{\pi}{2}t \\ \Rightarrow \left(-C_2 3^t - 3C_3 3^t\right)\sin\frac{\pi}{2}t + \left(3C_2 3^t - C_3 3^t\right)\cos\frac{\pi}{2}t &= 3^t \sin\frac{\pi}{2}t. \end{aligned}$$

Equating the like terms, we have

$$-C_2 - 3C_3 = 1,$$

$$\textit{and} \quad 3C_2 - C_3 = 0.$$

$$\implies C_3 = \frac{-3}{10} \textit{ and } C_2 = -\frac{1}{10}.$$

$$\textit{Thus, } y(t) = C_1 - \frac{3^t}{10}\left(\sin\frac{\pi}{12}t + 3\cos\frac{\pi}{2}t\right),$$

where C_1 is arbitrary.

Method of solving system of Linear Difference Equations

Consider the system of linear difference equations

$$L(E)y(t) + M(E)z(t) = r(t)$$

$$P(E)y(t) + Q(E)z(t) = s(t),$$

where $y(t)$ and $z(t)$ are the unknowns and L, M, P and Q are polynomials. Apply $Q(E)$ to the first equation and $M(E)$ to the second equation and then subtract to obtain

$$(Q(E)L(E) - M(E)P(E))y(t) = Q(E)r(t) - M(E)s(t).$$

This is a linear equation with constant coefficients. From the we can find $y(t)$ and then by substituting the expression for $y(t)$ in any one of the original equations, we get $z(t)$.

Example 2.3.8. *Solve the system*

$$y(t+2) - 3y(t) + z(t+1) - z(t) = 5^t$$

$$y(t+1) - 3y(t) + z(t+1) - 3z(t) = 2.5^t.$$

The given system of equations can be written as

$$\left(E^2 - 3\right)y(t) + (E-1)z(t) = 5^t \qquad (2.23a)$$

$$(E-3)y(t) + (E-3) \quad z(t) = 2.5^t. \qquad (2.23b)$$

Applying $(E-3)$ to the 1^{st} equation and $(E-1)$ s to the 2^{nd} equation, we get

$$\left(E^2 - 3\right)(E-3)y(t) + (t-3)(E-1)z(t) = (E-3)5^t$$

$$(E-1)(E-3)y(t) + (E-3)(E-1)z(t) = (E-1)25^t.$$

Subtracting, we get

$$\begin{aligned}
\left(\left(E^2-3\right)(E-3)-(E-1)(E-3)\right)y(t) &= (E-3)5^t-(E-1)2\cdot 5^t \\
(E-3)\left(E^2-E-2\right)y(t) &= (E-3-2E+2)5^t \\
(E-3)(E-2)(E+1)y(t) &= -(E+1)5^t \\
&= -\left(5^{t+1}+5^t\right) \\
&= -6\cdot 5^t. \qquad (2.24)
\end{aligned}$$

By the annihilator method, we can get an appropriate trial solution $y(t) = C5^t$. *Substituting this in* (2.24)*, we get*

$$\begin{aligned}
(E-3)(E-2)(E+1)C.5^t &= -6\cdot 5^t \\
\left(E^3-E^2+E+6\right)C.5^t &= -65^t \\
(5^{t+3}-4\cdot 5^{t+2}+5^{t+1}+6\cdot 5^t &= -6\cdot 5 \\
C\cdot 5^t\left(5.-4\cdot 5^2+5+6\right) &= -6\cdot 5^t \\
(125-100+11)c &= -6 \\
36C &= -6 \\
C &= -1/6
\end{aligned}$$

$$y(t) = C_1 3^t + C_2 2^t + C_3(-1)^2 = 5t.$$

Substituting this in (2.23b)*, we get:*

$$\begin{aligned}
(E-3)\left(C_1 3^t + C_{22}{}^t + C_3(-1)^t - 5^t\right) + (t-3)z(t) = 2\cdot 5^t \\
\Rightarrow C_1 3^{t+1} + C_2 2^{t+1} + C_3(t-1)^{t+1} - \frac{5^{t+1}}{6} - C_1 3^{t+1} - 3C_2 2^t \\
+3C_3(-1)^{t+1} + \frac{5t}{2} + (E-3)z(t) = 2\cdot 5^t \\
\Rightarrow C_2 2^t(2-3) + 4C_3(-1)^{t+1} + \frac{5^t}{2}(1-5/3) + (E-3)3(t) = 2\cdot 5^t \\
\Rightarrow -C_2 2^t + 4C_3(-1)^{t+1} - \frac{5}{3} + (E-3)r(t) = 2\cdot 5t \\
\therefore (E-3)z(t) = C_2 2^t + 4C_3 - (-1)^t + \frac{5^t}{3} + 2\cdot 5 = C_2 2^t + 4C_3(-1)^t + \frac{1}{3}.
\end{aligned}$$

Let us again use the annihilator method to find $z(t)$:

$$\begin{aligned}
(E-3)C2^t = C_2 2^t \Rightarrow C(2^{t+1} - 3\cdot 2^t) &= C_2 \cdot 2^t \\
\Rightarrow C &= -C_2(E-3)z(t) \\
&= -C_2 2^t \\
z(t) &= -C_2 2^t
\end{aligned}$$

$$\begin{aligned}
(E-3)C(-1)^t &= 4C_3(-1)^t \Rightarrow C((-1)^{t+1} - 3(-1)^t) = 4C_3(-1)^t \\
\Rightarrow -4(-1)^t &= 4C_3(-1)^t \\
C &= -C_3 \\
\text{and } \ (E-3)C\cdot 5^t = \frac{7}{3}5^t \Rightarrow C\left(5^{t+1} - 3\cdot 5^t\right) &= \frac{7}{3}5^t \\
\Rightarrow C(5-3) &= \frac{7}{3} \\
\Rightarrow C &= \frac{7}{6} \\
\text{Thus, } z(t) &= -C_2 2^t - C_3(-1)^t + \frac{7}{6}5^t + C_4 3^t.
\end{aligned}$$

Now, substituting the expressions for y and z in (1), we get

$$\begin{aligned}
(E^2-3)y(t) + (E-1)z(t) &= 5^t \\
\Longrightarrow \left(C_1 3^{t+2} + C_2 2^{t+2} + C_3(-1)^{t+2} - \frac{5^t}{6} \ -3C_1 3^t - 3C_2 2^t - 3C_3(-1)^t + \frac{5^t}{2}\right) + & \\
(-C_2 2^{t+1} - C_3(-1)^{t+1} + \frac{7}{6}5^{t+1} + C_4 3^{t+1} + C_2 2^t + C_3(-1)^t - \frac{7}{6}5^t - C_4 3^t) &= 5^t
\end{aligned}$$

$$\begin{aligned}
\Longrightarrow \left(6C_1 3^t + C_2 2^t - 2C_3(-1)^t - \frac{11}{3}5^t - C_2 2^t + 2C_3(-1)^t \ + \frac{14}{3}5^t + 2C_4 - 3t\right) &= 5^t \\
\Longrightarrow \ (bC_6 + 2C_4)\, 3tC_5^t &= 5^t \\
\Longrightarrow \ (6C_1 + 2C_4)\, 3^t + 5^t &= 0 \\
\Longrightarrow \ (6C_1 + 2C_4)\, 3^t &= 0 \\
\Longrightarrow \ C_4 &= -3C_1.
\end{aligned}$$

$\therefore$ *The general solution is*

$$\begin{aligned}
y(t) &= C_1 3^t + C_2 2^t + C_3(-1)^t - \frac{5^t}{6}, \\
z(t) &= -3C_1 3^t - C_2 2^t - C_3(-1)^t + \frac{7}{6}5^t.
\end{aligned}$$

The method of variation of parameters

If we assume that n linearly independent solutions of (2.9) are known, then the method of variations of parameters gives all solutions of equation (2.8) in terms of n indefinite terms. Let us do this for $n = 2$.

Let $u_1(t)$, $u_2(t)$ be independent solutions of

$$p_2(t)u(t+2) + p_1(t)u(t+1) + p_0(t)u(t) = 0. \tag{2.25}$$

We have to find a solution of

$$p_2(t)y(t+2) + p_1(t)y(t+1) + p_0(t)y(t) = r(t) \tag{2.26}$$

of the form

$$y(t) = a_1(t)u_1(t) + a_2(t)u_2(t),$$

where a_1 and a_2 are to be determined.

$$\begin{aligned}
\text{Then } y(t+1) &= a_1(t+1)u_1(t+1) + a_2(t+1)u_2(t+1) \\
&= a_1(t+1)u_1(t+1) + a_1(t)u_1(t+1) - a_1(t)u_1(t+1) \\
&\quad -a_2(t+1)u_2(t+1) + a_2(t)u_2(t+1) \\
&= a_2(t)u_2(t+1) - a_1(t)u_1(t+1) + a_2(t)u_2(t+1) \\
&\quad +\Delta a_1(t)u_1(t+1) + \Delta a_2(t)u_2(+t+1).
\end{aligned}$$

Choose $a_1(t)$ and $a_2(t)$ so that

$$\Delta a_1(t)u_1(t+1) + \Delta a_2(t)u_2(t+1) \quad = \quad 0. \tag{2.27}$$

$$\begin{aligned}
\therefore y(t+1) &= a_1(t)u_1(t+1) + a_2(t)u_2(t+1) \\
\text{Then} \quad y(t+2) &= a_1(t+1)u_1(t+2) + a_2(t+1)u_2(t+2) \\
&= a_1(t+1)u_1(t+2) + a_1(t)u_1(t+2) - a_1(t)u_1\,(t+2) + a_2(t+1)u_2(t+2) \\
&\quad +a_2(t)u_2(t+2) - a_2(t)u_2(t+2) \\
&= a_1(t)u_1(t+2) + a_2(t)u_2(t+2) + \Delta a_1(t)u_1(t+2) + \Delta a_2(t)u_2(t+2).
\end{aligned}$$

Substituting the expressions for $y(t), y(t+1)$ and $y[t+2\,)$ in equation (2.26), we get

$$\begin{aligned}
p_2(t)y(t+2)+p_1(t)y(t+1)+p_0(t)y(t) &= p_2(t)\left[a_1(t)u_1(t+2)+a_2(t)u_2(t+2)+\Delta a_1(t)u_1(t+2)\right.\\
&\quad \left.+\Delta a_2(t)u_2(t+2)\right]+p_1(t)\left[a_1(t)u_1(t+1)+a_2(t)u_2(t+1)\right]\\
&\quad +p_0(t)\left[a_1(t)u_1(t)+a_2(t)u_2(t)\right]\\
&= a_1(t)\left\{p_2(t)u_1(t+2)+p_1(t)u_1(t+1)+p_0(t)u_1(t)\right]\\
&\quad +a_2(t)\left[p_2(t)u_2(t+2)+p_1(t)u_2(t+1)+p_0(t)u_2(t)\right]\\
&\quad +p_2(t)\left[u_1(t+2)\Delta a_1(t)+u_2(t+2)\Delta a_2(t)\right]\\
&= p_2(t)\left[u_1(t+2)\Delta a_1(t)+u_2(t+2)\Delta a_2(t)\right].
\end{aligned}$$

Thus, $y(t)$ satisfies equation (2.26) if

$$u_1(t+2)\Delta a_1(t)+u_2(t+2)a_2(t)=\frac{r(t)}{p_2(t)}. \tag{2.28}$$

Therefore, $y(t)=a_1(t)u_1(t)+a_2(t)u_2(t)$ is a solution of equation (2.26) if $\Delta a_1(t), \Delta a_2(t)$ satisfy the linear equations (2.27) and (2.28). The determinant of the coefficients in equations (2.27) and (2.28) is

$$\begin{aligned}
&= \begin{vmatrix} u_1(t+1) & u_2(t+1) \\ u_1(t+2) & u_2(t+2) \end{vmatrix}\\
&= w(t+1)\\
&\neq 0 \quad (\text{Since } \; u_1, u_2 \text{ are linearly independent}).
\end{aligned}$$

$\therefore$ This system of equations (2.27) and (2.28) has a unique solution.

Example 2.3.9. *Find all solutions of* $\;y(t+2)-7y(t+1)+6y(t)=t$ *if* $u_1(t)=1$ *and* $u_2(t)=6^t$ *are the two independent solutions of its homogeneous equation.*

By variation of parameters method,

$$y(t)=a_1(t)u_1(t)+a_2(t)u_2(t)$$

is a solution of the given difference equation if

$$u_1(t+1)\Delta a_1(t)+u_2(t+1)\Delta a_2(t)=0$$

and

$$u_1(t+2)\Delta a_1(t)+u_2(t+2)\Delta a_2(t)=\frac{r(t)}{p_2(t)}.$$

Then, we have

$$\Delta a_1(t) + b^{t+1}\Delta a_2(t) = 0 \to (1)$$

$$\underline{\Delta a_1(t) + b^{t+2}\Delta a_2(t) = t \to (2)}$$

$$\text{(2) - (1)} \implies \Delta 6^{t+1(6-1)}\Delta a_2(t) = t$$

$$\Rightarrow \Delta a_2(t) = \frac{t}{56^{t+1}} = \frac{t}{30}6^{-t}$$

$$\therefore \Delta a_1(t) = -6^{t+1}\frac{t}{30}6^{t} = \frac{-t}{5}.$$

Then

$$\begin{aligned}
a_1(t) &= \sum\left(-\frac{t}{5}\right) + C \\
&= -\frac{1}{5}\sum t + C \\
&= -\frac{1}{5}\sum t^{(1)} + C \\
&= -\frac{1}{5}\sum \frac{t^{(2)}}{2} + C \\
&= -\frac{t^{(2)}}{10} + C \\
&= -\frac{t(t-1)}{10} + C.
\end{aligned}$$

Next,

$$\begin{aligned}
\Delta a_2(t) = \frac{t}{30}6^{-t} \Rightarrow a_2(t) &= \sum \frac{t}{30}6^{-t} + D \\
&= \frac{1}{30}\sum t\left(\frac{1}{6}\right)_i^t + D \\
&= \frac{1}{30}\left[t\left(\frac{-6}{5}\right)\left(\frac{1}{6}\right)^t - \sum\left(\frac{-6}{5}\right)\left(\frac{1}{6}\right)^{t+1}\right] + D \\
&= \frac{1}{30}\left[-\frac{6}{5}t + \left(\frac{1}{6}\right)^t + \frac{6}{5}\times\frac{1}{6}\times\left(\frac{-6}{5}\right)\left(\frac{1}{6}\right)^t\right] + D \\
&= \frac{-t}{25}\left(\frac{1}{6}\right)^t - \frac{1}{125}\left(\frac{1}{6}\right)^t + D.
\end{aligned}$$

∴ *The general solution of the given difference equation is*

$$\begin{aligned} y(t) &= a_1(t)u_1(t) + a_2(t)u_2(t) \\ &= \left(-\frac{t(t-1)}{10} + C\right) + \left[\frac{-t}{25}\left(\left(\frac{1}{6}\right)^t\right) - \frac{1}{125}\left(\left(\frac{1}{6}\right)^t\right) + D\right]6^t \\ &= C + D6^t - \frac{t^2}{10} + \frac{t}{10} - \frac{t}{25} - \frac{1}{125} \\ &= C + D6^t - \frac{t^2}{10} + \frac{3t}{50} - \frac{1}{125} \\ &= F + D6^t - \frac{t^2}{10} + \frac{3t}{50}, \end{aligned}$$

where F, D *are constants.*

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Annihilator method
2. Solving system of linear difference equations with constant coefficients
3. Method of variation of parameters

Check your Progress:

1. The characteristic polynomial of $u(t+n) + p_{n-1}u(t+n-1) + \cdots + p_0u(t) = 0$ is of the form ....

   (A) $\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0$ (B) $p_{n-1}\lambda^{n-1} + \cdots + p_0$

   (C) $p_{n-1} + p_{n-2} + \cdots + p_0$ (D) None of these

2. $\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0 = 0$ is the characteristic equation for the ....

   (A) n^{th} order equation (B) $(n-1)^{th}$ order equation

   (C) $(n+1)^{th}$ order equation (D) None of these

3. The annihilator of the difference equation $y(t+2) - 7y(t+1) + 6y(t) = t$ is

   (A) $(E-6)(E-1)$ (B) $(E+1)(E-6)$ (C) $(E-6)(E-1)^2$ (D) $(E-1)^2$

Unit Summary:

In this unit, the basic theory for linear difference equations is developed, and methods of finding closed form solutions to linear difference equations with constant coefficients are discussed.

Glossary:

- $W(t)$ - The matrix of Casorati
- $w(t)$ - Casoratian (The determinant of matrix of Casorati)

Self-Assessment Questions:

1. Find all solutions:

   (a) $u(t+1) - e^{3t}u(t) = 0$.

   (b) $u(t+1) - e^{\cos 2t}u(t) = 0$.

2. What is the order of this equation

$$\Delta^3 y(t) + \Delta^2 y(t) - \Delta y(t) - y(t) = 0?$$

3. Show that $u_1(t) = 2^t$ and $u_2(t) = 3^t$ are linearly independent solutions of

$$u(t+2) - 5u(t+1) + 6u(t) = 0$$

4. Solve $u(t+2) + 6u(t+1) + 3u(t) = 0$.

Exercises:

1. Suppose $y(1) = 2$ and find the solution of

$$y(t+1) - 3y(t) = e^t \quad (t = 1, 2, 3, \cdots)$$

2. (a) Show that $u_1(t) = t^2 + 2, u_2(t) = t^2 - 3t$ and $u_3(t) = 2t - 1$ are solutions of $\Delta^3 u(t) = 0$

   (b) Compute the Casoratian of the functions in (a) and determine whether they are linearly independent.

3. Solve by the annihilator method

$$y(t+2) + 4y(t) = \cos t.$$

4. Find all $u(t)$ and $v(t)$ that satisfy

$$u(t+2) - 3u(t) + 2v(t) = 0$$
$$u(t) + v(t+2) - 2v(t) = 0.$$

5. Use variation of parameters to solve

$$y(t+3) - 2y(t+2) - y(t+1) + 2y(t) = 8 \cdot 3^t.$$

Answers for check your progress:

Section 2.1	1. (B)	2. (B)	3. (C)
Section 2.2	1. (B)	2. (A)	3. (D)
Section 2.3	1. (A)	2. (A)	3. (D)

References:

1. W.G. Kelley and A.C. Peterson, "Difference Equations", 2nd Edition, Academic Press, New York, 2001.

Suggested Reading:

1. R.P. Agarwal, "Difference Equations and Inequalities", 2nd Edition, Marcel Dekker, New York, 2000.

2. S.N. Elaydi, "An Introduction to Difference Equations", 3rd Edition, Springer, India, 2008.

3. R. E. Mickens, "Difference Equations", 3rd Edition, CRC Press, 2015.

UNIT 3

Unit 3

Linear Difference Equations (continued)

Objectives:

This unit deals with solving linear difference equations with variable coefficients using generating functions and z-transforms.

3.1 Equations with Variable Coefficients

Lemma 3.1.1. *Let $u_1(t), u_2(t), \cdots, u_n(t)$ be solutions of the equation*

$$p_n(t)u(t+n) + p_{n-1}(t)u(t+n-1) + \cdots + p_0(t)u(t) = 0 \tag{3.1}$$

and let $w(t)$ be the corresponding Casoratian. Then $w(t)$ satisfies

$$w(t+1) = (-1)^n \frac{p_0(t)}{p_n(t)} w(t). \tag{3.2}$$

Proof. Given that $u_1(t), u_2(t), \cdots, u_n(t)$ are solutions of (3.1).

Then, we have

$$w(t+1) = \begin{vmatrix} u_1(t+1) & u_2(t+1) & \cdots & u_n(t+1) \\ u_1(t+2) & u_2(t+2) & \cdots & u_n(t+2) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(t+n-1) & u_2(t+n-1) & \cdots & u_n(t+n-1) \\ u_1(t+n) & u_2(t+n) & \cdots & u_n(t+n) \end{vmatrix}.$$

Since $w(t+1)$ is unchanged if we replace the last row by

$$\left(n^{\text{th}} \text{ row }\right) + \frac{p_1}{p_n} \times \left(1^{\text{st}} \text{ row }\right) + \cdots + \frac{p_{n-1}}{p_n} \times \left((n-1)^{\text{th}} \text{ row }\right),$$

we get,

$$w(t+1) = \begin{vmatrix} u_1(t+1) & u_2(t+1) & \cdots & u_n(t+1) \\ u_1(t+2) & u_2(t+2) & \cdots & u_n(t+2) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(t+n-1) & u_2(t+n-1) & \cdots & u_n(t+n-1) \\ u_1(t+n) + \frac{p_1}{p_n}u_1(t+1) & u_2(t+n) + \frac{p_1}{p_n}u_2(t+1) & \cdots & u_n(t+n) + \frac{p_1}{p_n}u_n(t+n-1) \\ \cdots + \frac{p_{n-1}}{p_n}u_1(t+n-1) & \cdots + \frac{p_{n-1}}{p_n}u_2(t+n-1) & \cdots & +\frac{p_{n-1}}{p_n}u_n(t+1) \end{vmatrix}$$

$$\implies w(t+1) = \begin{vmatrix} u_1(t+1) & u_2(t+1) & \cdots & u_n(t+1) \\ u_1(t+2) & u_2(t+2) & \cdots & u_n(t+2) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(t+n-1) & u_2(t+n-1) & \cdots & u_n(t+n-1) \\ -\frac{p_0}{p_n}u_1(t) & -\frac{p_0}{p_n}u_2(t) & \cdots & -\frac{p_0}{p_n}u_n(t) \end{vmatrix}$$

Rearranging the rows, we get

$$w(t+1) = (-1)^{n-1} \begin{bmatrix} -\frac{p_0}{p_n}u_1(t) & -\frac{p_0}{p_n}u_2(t) & \cdots & -\frac{p_0}{p_n}u_n(t) \\ u_1(t+1) & u_2(t+1) & \cdots & u_n(t+1) \\ u_1(t+2) & u_2(t+2) & \cdots & u_n(t+2) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(t+n-1) & u_2(t+n-1) & \cdots & u_n(t+n-1) \end{bmatrix}.$$

$$\implies w(t+1) = (-1)^n \frac{p_0(t)}{p_n(t)} \begin{bmatrix} u_1(t) & u_2(t) & \cdots & u_n(t) \\ u_1(t+1) & u_2(t+1) & \cdots & u_n(t+1) \\ \vdots & \vdots & \vdots & \vdots \\ u_1(t+n-1) & u_2(t+n-1) & \cdots & u_n(t+n-1) \end{bmatrix}$$

$$\implies w(t+1) = (-1)^n \frac{p_0(t)}{p_n(t)} w(t).$$

□

Theorem 3.1.2. ***Reduction of order method for a 2^{nd} order equation***

If $u_1(t)$ is a solution of

$$p_2(t)u(t+2) + p_1(t)u(t+1) + p_0(t)u(t) = 0 \tag{3.3}$$

that is never zero and $p_0(t)$ and $p_2(t)$ are not zero, then

$$u_2(t) = u_1(t) \sum \frac{w(t)}{u_1(t)u_1(t+1)}$$

yields an independent solution of equation (3.3), where $w(t)$ is a non-zero solution of equation $w(t+1) = (-1)^n \frac{p_0(t)}{p_n(t)} w(t)$.

Proof. Given that $u_1(t)$ is one of the solution of equation (3.3). Let $u_2(t)$ be the other solution of equation (3.3). We know that

$$\begin{aligned}
\Delta\left(\frac{u_2(t)}{u_1(t)}\right) &= \frac{u_1(t)\Delta u_2(t) - u_2(t)\Delta u_1(t)}{u_1(t)Eu_1(t)} \\
&= \frac{w(t)}{u_1(t)u_1(t+1)} \\
\Rightarrow \frac{u_2(t)}{u_1(t)} &= \sum \frac{w(t)}{u_1(t)u_1(t+1)} \\
\Longrightarrow \quad u_2(t) &= u_1(t)\sum \frac{w(t)}{u_1(t)u_1(t+1)}.
\end{aligned}$$

□

Example 3.1.3. *Solve the equation*

$$u(t+2) - u(t+1) - \frac{1}{t+1}u(t) = 0.$$

We know that, $u_1(t) = t+1$ *is a solution. By lemma (3.1), the Casoratian* $w(t)$ *satisfies,*

$$\begin{aligned}
w(t+1) &= (-1)^2\frac{p_0(t)}{p_n(t)}w(t) \\
&= \left(\frac{-1}{t+1}\right)w(t). \\
\Longrightarrow w(t) &= w(a)\prod_{s=a}^{t-1} p(s) \\
&= w(0)\prod_{s=0}^{t-1}\left(\frac{-1}{s+1}\right) \\
&= w(0)\bigg(-1\bigg)\left(\frac{-1}{2}\right)\cdots\left(\frac{-1}{t}\right) \\
&= w(0)\frac{(-1)^t}{t!} \\
&= \frac{(-1)^t}{t!} \text{ if } w(0) = 1
\end{aligned}$$

By Theorem (3.1.2), the second independent solution is given by

$$\begin{aligned}
u_2(t) &= u_1(t)\sum \frac{w(t)}{u_1(t)u_1(t+1)} \\
&= (t+1)\sum \frac{(-1)^t}{t!(t+1)(t+2)} \\
&= (t+1)\sum \frac{(-1)^t}{(t+2)!} \\
\Longrightarrow u_2(t) &= (t+1)\sum_{k=0}^{t-1} \frac{(-1)^k}{(k+2)!}
\end{aligned}$$

Hence, the general solution is,

$$\begin{aligned}
u(t) &= Cu_1(t) + Du_2(t) \\
&= C(t+1) + D(t+1)\sum_{k=0}^{t-1} \frac{(-1)^k}{(k+2)!} \\
\implies u(t) &= (t+1)\left[C + D\sum_{k=0}^{t-1} \frac{(-1)^k}{(k+2)!}\right]
\end{aligned}$$

where C and D are constants.

Example 3.1.4. *Solve*

$$t(t+1)\Delta^2 u(t) + at\Delta u(t) + bu(t) = 0, \tag{3.4}$$

where a and b are constants.

(The Equation (3.4) is similar to the Cauchy-Euler differential equation.)

By substituting the trial solution $u(t) = (t+r-1)^{\underline{r}}$, we have

$$t(t+1)\Delta^2 (t+r-1)^{\underline{r}} + at\Delta(t+r-1)^{\underline{r}} + b(t+r-1)^{\underline{r}} = 0$$

$$t(t+1)r(r-1)(t+r-1)^{\underline{r-2}} + atr(t+r-1)^{\underline{r-1}} + b(t+r-1)^{\underline{r}} = 0 \tag{3.5}$$

Now,

$$\begin{aligned}
t(t+r-1)^{\underline{r-1}} &= t(t+r-1)(t+r-1-1)(t+r-1-2)\dots \\
&\quad (t+r-1-(r-1-1)) \\
&= t(t+r-1)(t+r-2)\dots(t+2)(t+1) \\
&= (t+r-1)(t+r-2)\dots(t+2)(t+1)t \\
&= (t+r-1)^{\underline{r}} \qquad (3.6a)
\end{aligned}$$

$$\begin{aligned}
\text{and} \quad t(t+1)(t+r-1)^{\underline{r-2}} &= t(t+1)(t+r-1)(t+r-2)\dots \\
&\quad (t+r-1-(r-4))(t+r-1-(r-3)) \\
&= t(t+1)(t+r-1)\dots(t+3)(t+2) \\
&= (t+r-1)(t+r-2)\cdots(t+2)(t+1)t \\
&= (t+r-1)^{\underline{r}} \qquad (3.6b)
\end{aligned}$$

Substituting (3.6a) *and* (3.6a) *in* (3.5), *we get*

$$\begin{aligned} r(r-1)(t+r-1)^{\underline{r}} + ar(t+r-1)^{\underline{r}} + b(t+r-1)^{\underline{r}} &= 0 \\ \implies r^2(t+r-1)^{\underline{r}} + (a-1)r(t+r-1)^{\underline{r}} + b(t+r-1)^{\underline{r}} &= 0 \\ \implies \left(r^2 + (a-1)r + b\right)(t+r-1)^{\underline{r}} = 0 \\ \implies r^2 + (a-1)r + b = 0 \end{aligned} \tag{3.7}$$

If equation (3.7) *has distinct roots* r_1, r_2, *then the difference equation has the independent solutions*

$$u_1(t) = (t+r_1-1)^{\underline{r_1}},$$
$$u_2(t) = (t+r_2-1)^{\underline{r_2}}.$$

In the case of repeated roots, we can use Theorem 3.1.2 *to obtain the second solution. Now, taking* $a=-5$ *and* $b=9$ *in equation* (3.4), *we have*

$$t(t+1)\Delta^2 u(t) - 5t\Delta u(t) + 9u(t) = 0. \tag{3.8}$$

Then, equation (3.7) *becomes*

$$\begin{aligned} r^2 - 6r + 9 &= 0 \\ \implies (r-3)^2 &= 0 \\ \implies r &= 3. \end{aligned}$$

So, we get one of the solution of (3.8) *as*

$$\begin{aligned} u_1(t) &= (t+r-1)^{\underline{r}} \\ &= (t+3-1)^{\underline{r}} = (t+2)^{\underline{3}} \\ &= (t+2)(t+1)t. \end{aligned}$$

Next, let us rewrite the equation (3.8) *as*

$$\begin{aligned} t(t+1)(E-I)^2u(t) - 5t(E-I)u(t) + 9u(t) = 0 \\ \implies t(t+1)u(t+2) - \left(2t^2+7t\right)u(t+1) + (t+3)^2u(t) &= 0. \end{aligned}$$

Here $n=2$, $p_0(t) = (t+3)^2$, $p_2(t) = t(t+1)$.
Then, $w(t)$ *satisfies* $w(t+1) = (-1)^n \frac{p_0(t)}{p_n(t)} w(t)$.

$$\implies w(t+1) = (-1)^2 \frac{(t+3)^2}{t(t+1)} w(t)$$

$$\implies w(t+1) = \frac{(t+3)^2}{t(t+1)} w(t)$$

Then,

$$\begin{aligned} w(t) &= w(1) \prod_{s=1}^{t-1} \frac{s+3^2}{s(s+1)} \\ &= w(1) \left(\frac{4^2}{1(2)}\right)\left(\frac{5^2}{2(3)}\right) \cdots \left(\frac{(t+2)^2}{(t-1)(t)}\right) \\ &= w(1) \frac{(t+2)!(t+2)!}{(t-1)!t!.3^2.2^2} \\ &= w(1) \frac{(t+2)(t+1)t(t+2)(t+1)}{3^2.2^2} \\ &= t(t+1)^2(t+2)^2 \quad if \quad w(1) = 3^2.2^2. \end{aligned}$$

Thus, the second solution of (3.8) is given by

$$\begin{aligned} u_2(t) &= u_1(t) \sum \frac{w(t)}{u_1(t)u_1(t+1)} \\ &= (t+2)(t+1)t \sum \frac{t(t+1)^2(t+2)^2}{(t+2)(t+1)t(t+3)(t+2)(t+1)} \\ &= (t+2)(t+1)t \sum \frac{1}{t+3} \\ &= (t+2)^{\underline{3}} \sum \frac{1}{t+3}. \end{aligned}$$

$\therefore$ *The general solution is*

$$\begin{aligned} u(t) &= Cu_1(t) + Du_2(t) \\ &= C(t+2)^{\underline{3}} + D(t+2)^{\underline{3}} \sum \frac{1}{t+3} \\ &= (t+2)^{\underline{3}} \left[C + D \sum \frac{1}{t+3}\right]. \end{aligned}$$

where C, D are constants.

Example 3.1.5. *Solve*

$$(n+2)u_{n+2} - (n+3)u_{n+1} + 2u_n = 0$$

where $n = 0, 1, 2, \cdots$.

Let the generating function be

$$g(x) = \sum_{n=0}^{\infty} u_n x^n.$$

First, multiplying each term in the difference equation by x^n and summing as n goes from 0 to ∞, we get

$$\sum_{n=0}^{\infty}(n+2)u_{n+2}x^n - \sum_{n=0}^{\infty}(n+3)u_{n+1}x^n + 2\sum_{n=0}^{\infty}u_nx^n = 0.$$

Next, making a change of index in the first two summations, we get,

$$\sum_{n=2}^{\infty}nu_nx^{n-2} - \sum_{n=1}^{\infty}(n+2)u_nx^{n-1} + 2\sum_{n=0}^{\infty}u_nx^n = 0. \tag{3.9}$$

Since $g'(x) = \sum_{n=1}^{\infty} nu_nx^{n-1}$, we can write

$$g'(x) = u_1 + \sum_{n=2}^{\infty}nu_nx^{n-1}$$

$$\implies \sum_{n=2}^{\infty}nu_nx^{n-2} = \frac{1}{x}\left(g'(x) - u_1\right)$$

and

$$g(x) = \sum_{n=0}^{\infty}u_nx^n = u_0 + \sum_{n=1}^{\infty}u_nx^n$$

$$\implies \sum_{n=1}^{\infty}u_nx^{n-1} = \frac{1}{x}\left(g(x) - u_0\right)$$

Substituting these expressions into (3.9), we have

$$\frac{1}{x}\left(g'(x) - u_1\right) - g'(x) - \frac{2}{x}\left(g(x) - u_0\right) + 2g(x) = 0$$

or

$$g'(x) - 2g(x) = \frac{u_1 - 2u_0}{1-x}.$$

For $u_1 = 2u_0$, we have $\quad g'(x) - 2g(x) = 0$

$$\implies g(x) = e^{2x}$$

$$\implies g(x) = \sum_{n=0}^{\infty}\frac{2^n}{n!}x^n$$

$$\therefore u_n = \frac{2^n}{n!}, (n = 0, 1, 2, \cdots).$$

This is one of the solution of the given equation.

To find the second solution, consider

$$w(n+1) = (-1)^2\frac{2}{n+2}w(n)$$

$$\begin{aligned}
\implies w(n) &= w(0)\prod_{s=0}^{n-1}\frac{2}{(s+2)} \\
&= w(0)\left(\frac{2}{2}\right)\left(\frac{2}{1+2}\right)\left(\frac{2}{2+2}\right)\cdots\left(\frac{2}{1+n}\right) \\
&= w(0)\frac{2^n}{(n+1)!} \\
&= \frac{2^n}{(n+1)!} \quad if \quad w(0)=1.
\end{aligned}$$

$\therefore$ The second solution is given by

$$\begin{aligned}
v_n &= u_n\sum\frac{w(n)}{u_n u_{n+1}} \\
&= \frac{2^n}{n!}\sum\frac{\frac{2^n}{(n+1)!}}{\frac{2^n}{n!}\frac{2^{n+1}}{(n+1)!}} \\
&= \frac{2^n}{n!}\sum_{k=0}^{n-1}\frac{k!}{2^{k+1}}.
\end{aligned}$$

Hence, the general solution is $Cu_n + Dv_n$, where C, D are constants.

Example 3.1.6. *Find the factorial series solution of*

$$2u(t+2) + (t+2)(t+1)u(t+1) - (t+2)(t+1)u(t) = 0.$$

We can rewrite it as

$$2u(t+2) + (t+2)(t+1)\Delta u(t) = 0.$$

Substituting $u(t) = \sum_{k=0}^{\infty} a_k t^{\underline{-k}}$, we have

$$\sum_{k=0}^{\infty} 2a_k(t+2)^{\underline{-k}} + (t+2)(t+1)\sum_{k=1}^{\infty} a_k(-k)t^{\underline{-k-1}} = 0.$$

Since

$$\begin{aligned}
(t+2)(t+1)t^{\underline{-k-1}} &= (t+2)(t+1)\frac{\Gamma(t+1)}{\Gamma(t-(-k-1)+1)} \\
&= (t+2)(t+1)\frac{\Gamma(t+1)}{\Gamma(t+k+2} \\
&= (t+2)\frac{\Gamma(t+2)}{\Gamma(t+k+2)} \\
&= \frac{\Gamma(t+3)}{\Gamma(t+k+2)} \\
&= (t+2)^{\underline{-k+1}},
\end{aligned}$$

we have

$$\sum_{k=0}^{\infty} 2a_k(t+2)^{\underline{-k}} + \sum_{k=1}^{\infty} a_k(-k)(t+2)^{\underline{-k+1}} = 0.$$

Make the change of index $k \to k+1$ *in the second summation and combine the series to obtain*

$$\sum_{k=0}^{\infty} \left[2a_k - (k+1)a_{k+1}\right](t+2)^{\underline{-k}} = 0.$$

Then a_0 *is arbitrary and*

$$a_{k+1} = \frac{2}{k+1}a_k \quad (k \geq 0),$$

so

$$a_k = \frac{2^k}{k!}a_0.$$

$\therefore$ *A factorial series solution is*

$$u(t) = a_0 \sum_{k=0}^{\infty} \frac{2^k}{k!} t^{\underline{-k}},$$

and the series converges for all t *except the negative integers.*

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Solving linear difference equations with variable coefficients
2. Reduction of order method
3. Method of generating function
4. Factorial series solution

Check your Progress:

1. If $u_1(t), u_2(t), \cdots, u_n(t)$ are solutions of the equation $p_n(t)u(t+n) + p_{n-1}(t)u(t+n-1) + \cdots + p_0(t)u(t) = 0$, then their Casoratian $w(t)$ satisfies ....

   (A) $w(t+1) = (-1)^n \frac{p_n(t)}{p_0(t)} w(t)$ (B) $w(t+1) = (-1)^n \frac{p_0(t)}{p_n(t)} w(t)$

   (C) $w(t+1) = p_n(t)w(t)$ (D) None of these

2. Which of the following is a solution of $u(t+2) - u(t+1) - \frac{1}{t+1}u(t) = 0$?

(A) t (B) $t-1$ (C) $t+1$ (D) None of these

3. The order of the Cauchy – Euler equation is ....

(A) 1 (B) 2 (C) n (D) None of these

3.2 The z-Transform

The z-transform is a mathematical device similar to a generating function which provides an alternative method for solving linear difference equations as well as certain summation equations.

Definition 3.2.1. *The z-transorm of a sequence $\{y_k\}$ is a function Y(z) of a complex variable defined by*

$$Y(z) = Z(y_k) = \sum_{k=0}^{\infty} \frac{y_k}{z^k}$$

for those values of z for which the series converges. We say that the z-transform "exists" provided there is a number $R > 0$ such that $\sum_{k=0}^{\infty} \frac{y_k}{z^k}$ converges for $|z| > R$.

The sequence $\{y_k\}$ is said to be "exponentially bounded" if there is an $M > 0$ and a $c > 1$ such that

$$|y_k| \leq Mc^k$$

for $k \geq 0$,

Theorem 3.2.2. *If the sequence $\{y_k\}$ is exponentially bounded, then the z transform of $\{y_k\}$ exists.*

Proof. Assume that the sequence $\{y_k\}$ is exponentially bounded. Then there is an $M > 0$ and a $c > 1$ such that

$$|y_k| \leq Mc^k$$

for $k \geq 0$. We have

$$\sum_{k=0}^{\infty} \left|\frac{y_k}{z^k}\right| \leq \sum_{k=0}^{\infty} \frac{|y_k|}{|z|^k} \leq M \sum_{k=0}^{\infty} \left|\frac{c}{z}\right|^k$$

and the last sum converges for $|z| > c$. It follows that the z-transform of the sequence $\{y_k\}$ exists. □

In this section we will frequently use, without reference, the following theorem.

Theorem 3.2.3. *If the sequence $\{f_k\}$ is exponentially bounded, each solution of the n^{th} order difference equation*

$$y_{k+n} + p_1 y_{k+n-1} + p_2 y_{k+n-2} + \cdots + p_n y_k = f_k$$

is exponentially bounded and hence its z-transform exists.

Proof. We will give the proof of this theorem just for the case $n = 2$. Assume y_k is a solution of the second order equation

$$y_{k+2} + p_1 y_{k+1} + p_2 y_k = f_k$$

and $\{f_k\}$ is exponentially bounded. Since $\{f_k\}$ is exponentially bounded, there is an $M > 0$ and a $c > 1$ such that

$$|f_k| \leq Mc^k$$

for $k \geq 0$. Since y_k is a solution of the above second order difference equation, we have that

$$|y_{k+2}| \leq |p_1| \, |y_{k+1}| + |p_2| \, |y_k| + Mc^k. \tag{3.10}$$

Let

$$B = \max \{|p_1|, |p_2|, |y_0|, |y_1|, M, c\}$$

We now prove by induction that

$$|y_k| \leq 3^{k-1} B^k \tag{3.11}$$

for $k \geq 0$ ($k = 1, 2, 3 \cdots$). It is easy to see that the inequality (3.11)is true for $k = 1$. Now assume that $k_0 \geq 1$ and that the inequality (3.11) is true for $1 \leq k \leq k_0$. Letting $k = k_0 - 1$ in (3.10), we have that

$$|y_{k_0+1}| \leq |p_1| \, |y_{k_0}| + |p_2| \, |y_{k_0-1}| + Mc^{k_0-1}$$

Using the induction hypothesis and the definition of B we get that

$$|y_{k_0+1}| \leq B3^{k_0-1}B^{k_0} + B3^{k_0-2}B^{k_0-1} + BB^{k_0-1}$$

It follows that

$$|y_{k_0+1}| \leq 3^{k_0-1}B^{k_0+1} + 3^{k_0-1}B^{k_0+1} + 3^{k_0-1}B^{k_0+1} = 3^{k_0}B^{k_0+1}$$

which completes the induction. From the inequality (3.11),

$$|y_k| \leq (3B)^k$$

for $k = 1, 2, 3 \cdots$, so y_k is exponentially bounded. By Theorem (3.2.2) , the z transform of y_k exists. □

Example 3.2.4. *Find the z-transform of the sequence $\{y_k = 1\}$.*

The z-transform of the sequence $\{y_k = 1\}$ is given by

$$\begin{aligned}
Y(z) = Z(1) &= \sum_{k=0}^{\infty} \frac{1}{z^k} \\
&= 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \\
&= (1 - \frac{1}{z})^{-1}
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{1 - \frac{1}{z}} \\
&= \frac{z}{z-1} \quad where \quad \frac{1}{|z|} < 1 \\
\Longrightarrow Y(z) &= \frac{z}{z-1} \quad where \quad |z| > 1
\end{aligned}$$

Example 3.2.5. *Find the z-transform of the sequence $\{u_k = a^k\}$.*

The z-transform of the sequence $\{u_k = a^k\}$ is given by

$$\begin{aligned}
U(z) = Z(a^k) &= \sum_{k=0}^{\infty} \frac{a^k}{z^k} \\
&= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \dots \\
&= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \dots \\
&= (1 - \frac{a}{z})^{-1} \\
&= \frac{1}{1 - \frac{a}{z}} \\
&= \frac{z}{z-a} \quad where \quad \frac{|a|}{|z|} < 1 \\
U(z) &= \frac{z}{z-a} \quad where \quad |z| > |a|
\end{aligned}$$

Example 3.2.6. *Find the z-transform of the sequence $\{v_k = k\}_{k=0}^{\infty}$.*

The z-transform of the sequence $\{v_k = k\}_{k=0}^{\infty}$ *is given by*

$$\begin{aligned} V(z) = Z(k) &= \sum_{k=0}^{\infty} \frac{k}{z^k} \\ &= \sum_{k=0}^{\infty} \frac{k+1}{z^{k+1}}. \end{aligned}$$

$$\begin{aligned} \Longrightarrow V(z) &= \frac{1}{z}\sum_{k=0}^{\infty} \frac{k+1}{z^k} \\ &= \frac{1}{z}\sum_{k=0}^{\infty} \frac{k}{z^k} + \frac{1}{z}\sum_{k=0}^{\infty} \frac{1}{z^k} \\ \Longrightarrow V(z) &= \frac{1}{z}V(z) + \frac{1}{z}Y(z) \quad \textit{where} \quad Y(z) = Z(1) \\ \Longrightarrow (1-\frac{1}{z})V(z) &= \frac{1}{z}Y(z) \quad \textit{where} \quad \frac{1}{|z|} < 1 \\ \Longrightarrow (\frac{z-1}{z})V(z) &= \frac{1}{z}Y(z) \quad \textit{where} \quad |z| > 1 \\ \Longrightarrow (z-1)V(z) &= \frac{z}{z-1} \quad \textit{where} \quad |z| > 1 \\ \Longrightarrow V(z) &= \frac{z}{(z-1)^2} \quad \textit{where} \quad |z| > 1. \end{aligned}$$

Theorem 3.2.7. ***Linearity Theorem***

If a and b are constants, then

$$Z(au_k + bv_k) = aZ(u_k) + bZ(v_k)$$

for those z in the common domain of U(z) and V(z).

Proof. Consider,

$$\begin{aligned} Z(au_k + bv_k) &= \sum_{k=0}^{\infty} \frac{au_k + bv_k}{z^k} \\ &= \sum_{k=0}^{\infty} \frac{au_k}{z^k} + \sum_{k=0}^{\infty} \frac{bv_k}{z^k} \\ &= a\sum_{k=0}^{\infty} \frac{u_k}{z^k} + b\sum_{k=0}^{\infty} \frac{v_k}{z^k} \\ \Longrightarrow Z(au_k + bv_k) &= aZ(u_k) + bZ(v_k). \end{aligned}$$

□

Example 3.2.8. *Find the z-transform of the sequence* $\{v_k = sinak\}_{k=0}^{\infty}$.

$$\begin{aligned}
Z(sinak) &= Z(\frac{e^{iak} - e^{-iak}}{2i}) \\
&= \frac{1}{2i}Z(e^{iak}) - \frac{1}{2i}Z(e^{-iak}) \\
&= \frac{1}{2i}\sum_{k=0}^{\infty}\frac{e^{iak}}{z^k} - \frac{1}{2i}\sum_{k=0}^{\infty}\frac{e^{-iak}}{z^k} \\
&= \frac{1}{2i}[1 + \frac{e^{ia}}{z} + (\frac{e^{ia}}{z})^2 + \dots] - \frac{1}{2i}[1 + \frac{e^{-ia}}{z} + (\frac{e^{-ia}}{z})^2 + \dots] \\
&= \frac{1}{2i}\Big[\frac{1}{1 - \frac{e^{ia}}{z}}\Big] - \frac{1}{2i}\Big[\frac{1}{1 - \frac{e^{-ia}}{z}}\Big] \\
&= \frac{1}{2i}\Big[\frac{z}{z - e^{ia}}\Big] - \frac{1}{2i}\Big[\frac{z}{z - e^{-ia}}\Big] \\
&= \frac{1}{2i}\Big[\frac{z}{z - e^{ia}} - \frac{z}{z - e^{-ia}}\Big] \\
&= \frac{1}{2i}\Big[\frac{z(z - e^{-ia}) - z(z - e^{ia})}{(z - e^{ia})(z - e^{-ia})}\Big] \\
&= \frac{1}{2i}\Big[\frac{z^2 - ze^{-ia} - z^2 + ze^{ia}}{z^2 - ze^{-ia} - ze^{ia} + e^{ia}e^{-ia}}\Big] \\
&= \frac{1}{2i}\Big[\frac{ze^{ia} - ze^{-ia}}{z^2 - z(e^{ia} + e^{-ia}) + 1}\Big]
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2i}\Big[\frac{z(e^{ia} - e^{-ia})}{z^2 - z(e^{ia} + e^{-ia}) + 1}\Big] \\
&= \frac{1}{2i}\Big[\frac{2izsina}{z^2 - 2zcosa + 1}\Big] \\
\implies Z(sinak) &= \frac{zsina}{z^2 - 2zcosa + 1}.
\end{aligned}$$

Example 3.2.9. *Find the z-transform of the sequence* $\{v_k = cosak\}_{k=0}^{\infty}$.

$$\begin{aligned}
Z(cosak) &= Z(\frac{e^{iak}+e^{-iak}}{2}) \\
&= \frac{1}{2}Z(e^{iak}) + \frac{1}{2i}Z(e^{-iak}) \\
&= \frac{1}{2}\sum_{k=0}^{\infty}\frac{e^{iak}}{z^k} + \frac{1}{2i}\sum_{k=0}^{\infty}\frac{e^{-iak}}{z^k} \\
&= \frac{1}{2}[1+\frac{e^{ia}}{z}+(\frac{e^{ia}}{z})^2+\dots] + \frac{1}{2i}[1+\frac{e^{-ia}}{z}+(\frac{e^{-ia}}{z})^2+\dots] \\
&= \frac{1}{2}\Big[\frac{1}{1-\frac{e^{ia}}{z}}\Big] + \frac{1}{2i}\Big[\frac{1}{1-\frac{e^{-ia}}{z}}\Big] \\
&= \frac{1}{2}\Big[\frac{z}{z-e^{ia}}\Big] + \frac{1}{2i}\Big[\frac{z}{z-e^{-ia}}\Big] \\
&= \frac{1}{2}\Big[\frac{z}{z-e^{ia}} + \frac{z}{z-e^{-ia}}\Big] \\
&= \frac{1}{2}\Big[\frac{z(z-e^{-ia})+z(z-e^{ia})}{(z-e^{ia})(z-e^{-ia})}\Big] \\
&= \frac{1}{2}\Big[\frac{z^2-ze^{-ia}+z^2-ze^{ia}}{z^2-ze^{-ia}-ze^{ia}+e^{ia}e^{-ia}}\Big] \\
&= \frac{1}{2}\Big[\frac{2z^2-ze^{ia}-ze^{-ia}}{z^2-z(e^{ia}+e^{-ia})+1}\Big] \\
&= \frac{1}{2}\Big[\frac{2z^2-z(e^{ia}+e^{-ia})}{z^2-z(e^{ia}+e^{-ia})+1}\Big] \\
&= \frac{1}{2}\Big[\frac{2z^2-2zcosa}{z^2-2zcosa+1}\Big] \\
\Longrightarrow Z(cosak) &= \frac{z^2-zcosa}{z^2-2zcosa+1}.
\end{aligned}$$

Theorem 3.2.10. *If* $Y(z) = Z(y_k)$ *for* $|z| > r$*, then*

$$Z((k+n-1)^{(n)}y_k) = (-)^n z^n \frac{d^nY}{dz^n}(z) \text{for } |z| > r.$$

Proof. By definition,

$$Y(z) = \sum_{k=0}^{\infty}\frac{y_k}{z^k} = \sum_{k=0}^{\infty} y_k z^{-k}$$

for $|z| > r$.

Differentiating (3.2) with respect to z, we get

$$
\begin{aligned}
\frac{d}{dz}Y(z) &= \sum_{k=0}^{\infty}(-k)y_k z^{-(k+1)}\\
\implies \frac{d^2}{dz^2}Y(z) &= \sum_{k=0}^{\infty}(k)(k+1)y_k z^{-(k+2)}\\
\implies \frac{d^3}{dz^3}Y(z) &= -\sum_{k=0}^{\infty}(k)(k+1)(k+2)y_k z^{-(k+3)}\\
&\vdots\\
\implies \frac{d^n}{dz^n}Y(z) &= (-)^n\sum_{k=0}^{\infty}(k)(k+1)\ldots(k+n-1)y_k z^{-(k+n)}\\
&= \frac{(-1)^n}{z^n}\sum_{k=0}^{\infty}\frac{(k+n-1)(k+n-2)\ldots(k+2)(k+1)k}{z^k}y_k\\
&= \frac{(-1)^n}{z^n}\sum_{k=0}^{\infty}\frac{(k+n-1)^{(n)}}{z^k}y_k\\
\implies \frac{d^n}{dz^n}Y(z) &= \frac{(-1)^n}{z^n}Z((k+n-1)^{(n)}y_k)\\
\implies Z((k+n-1)^{(n)}y_k) &= (-1)^n z^n\frac{d^nY}{dz^n}(z).
\end{aligned}
$$

□

Note: For $n=1$, we get the special case $Z(ky_k) = -zY'(z)$.

Example 3.2.11. *Find* $Z(ka^k)$.

$$
\begin{aligned}
Z(ky_k) &= -z\frac{d}{dz}Y(z)\\
&= -z\frac{d}{dz}Z(y_k)\\
\implies Z(ka^k) &= -z\frac{d}{dz}Z(a^k)\\
&= -z\frac{d}{dz}(\frac{z}{z-a})\\
&= -z\Big(\frac{(z-a)-z}{(z-a)^2}\Big)\\
&= -z\Big(\frac{-a}{(z-a)^2}\Big)\\
\implies Z(ka^k) &= \frac{az}{(z-a)^2}
\end{aligned}
$$

Example 3.2.12. *Find $Z(k^2)$.*

The given problem can be written as $Z(k^2) = Z(k \cdot k)$.

$$\begin{aligned} Z(ky_k) &= -z\frac{d}{dz}Y(z) \\ &= -z\frac{d}{dz}Z(y_k) \\ \implies Z(k^2) &= -z\frac{d}{dz}Z(k) \\ &= -z\frac{d}{dz}(\frac{z}{z-a}) \\ &= -z\frac{(z-1)^2 - 2(z-1)}{(z-1)^4} \end{aligned}$$

$$\begin{aligned} \implies Z(k^2) &= \frac{-z(z-1)((z-1)-2z)}{(z-1)^4} \\ &= \frac{z(z-1)(-1-z)}{(z-1)^4} \\ \implies Z(k^2) &= \frac{z(z+1)}{(z-1)^3} \end{aligned}$$

Definition 3.2.13. *Define the unit step sequence u(n) by*

$$u_k(n) = \begin{cases} 0 & \textit{if} \quad 0 \leq k \leq n-1 \\ 1 & \textit{if} \quad n \leq k \end{cases}$$

Note: The unit step sequence has a single step of unit height located at $k = n$.

Theorem 3.2.14. ***Shifting Theorem***

For n, a positive integer

$$\begin{aligned} Z(y_{k+n}) &= z^n Z(y_k) - \sum_{m=0}^{n-1} y_m z^{n-m}, \\ Z(y_{k-n}u_k(n)) &= z^{-n} Z(y_k). \end{aligned}$$

Proof. Consider

$$\begin{aligned} Z(y_{k+n}) &= \sum_{k=0}^{\infty} \frac{y_{k+n}}{z^k} \\ &= \sum_{k=0}^{\infty} y_{k+n} z^{-k} \\ &= \sum_{k=n}^{\infty} y_k z^{-k+n} \end{aligned}$$

$$
\begin{aligned}
\implies Z(y_{k+n}) &= z^n \sum_{k=n}^{\infty} y_k z^{-k} \\
&= z^n \Big[\sum_{k=0}^{\infty} y_k z^{-k} - \sum_{m=0}^{n-1} y_m z^{-m} \Big] \\
&= z^n \Big[\sum_{k=0}^{\infty} \frac{y_k}{z^k} - \sum_{m=0}^{n-1} y_m z^{-m} \Big] \\
&= z^n \Big[Z(y_k) - \sum_{m=0}^{n-1} y_m z^{-m} \Big] \\
\implies Z(y_{k+n}) &= z^n Z(y_k) - \sum_{m=0}^{n-1} y_m z^{n-m}.
\end{aligned}
$$

Now,

$$
\begin{aligned}
Z(y_{k-n} u_k(n)) &= \sum_{k=0}^{\infty} \frac{y_{k-n} u_k(n)}{z^k} \\
&= \sum_{k=0}^{\infty} y_{k-n} u_k(n) z^{-k} \\
&= \sum_{k=0}^{n-1} y_{k-n} u_k(n) z^{-k} + \sum_{k=n}^{\infty} y_{k-n} u_k(n) z^{-k} \\
&= \sum_{k=n}^{\infty} y_{k-n} u_k(n) z^{-k} \\
&= \sum_{k=n}^{\infty} y_{k-n} z^{-k} = \sum_{k=0}^{\infty} y_k z^{-k-n} \\
&= z^{-n} \sum_{k=0}^{\infty} \frac{y_k}{z^k} \\
\implies Z(y_{k-n} u_k(n)) &= z^{-n} Z(y_k).
\end{aligned}
$$

□

Example 3.2.15. *Find* $Z(u_k(n))$.

We know that $Z(y_{k-n}u_k(n)) = z^{-n}Z(y_k)$.

Therefore,

$$
\begin{aligned}
Z(1 \cdot u_k(n)) &= z^{-n} Z(1) \\
&= z^{-n} \sum_{k=0}^{\infty} \frac{1}{z^k} \\
&= z^{-n} \frac{z}{z-1} \\
\implies Z(u_k(n)) &= \frac{z^{1-n}}{z-1}
\end{aligned}
$$

Example 3.2.16. *Find* $Z(y_k)$ *if* $y_k = 2, 0 \leq k \leq 99; \quad y_k = 5, 100 \leq k.$

$$\begin{aligned}
Z(y_k) &= \sum_{k=0}^{\infty} \frac{y_k}{z^k} \\
&= \sum_{k=0}^{99} \frac{y_k}{z^k} + \sum_{k=100}^{\infty} \frac{y_k}{z^k} \\
&= \sum_{k=0}^{99} \frac{2}{z^k} + \sum_{k=100}^{\infty} \frac{5}{z^k} \\
&= 2\sum_{k=0}^{99} \Big(\frac{1}{z}\Big)^k + 5\sum_{k=100}^{\infty} \Big(\frac{1}{z}\Big)^k \\
&= 2\Big(\frac{(\frac{1}{z})^k}{\frac{1}{z}-1}\Big)_{k=0}^{100} + 5\Big(\frac{(\frac{1}{z})^k}{\frac{1}{z}-1}\Big)_{k=100}^{\infty} \\
&= 2\Big(\frac{(\frac{1}{z})^{100}z}{1-z} - \frac{z}{1-z}\Big) + 5\Big(\frac{(\frac{1}{z})^{\infty}z}{1-z} - \frac{(\frac{1}{z})^{100}z}{1-z}\Big)
\end{aligned}$$

$$\begin{aligned}
\implies Z(y_k) &= \frac{-3z(\frac{1}{z})^{100}}{1-z} - \frac{2z}{1-z} \\
&= \frac{3z}{z^{100}(z-1)} + \frac{2z}{z-1} \\
&= \frac{3}{z^{99}(z-1)} + \frac{2z}{z-1} \\
&= \frac{3+2zz^{99}}{z^{99}(z-1)} \\
\implies Z(y_k) &= \frac{3+2z^{100}}{z^{99}(z-1)}.
\end{aligned}$$

Theorem 3.2.17. *For any integer* $n \geq 0$*,*

$$Z((k+n-1)^{(n)}) = \frac{n!z^n}{(z-1)^{n+1}},$$

$$Z(k^{(n)}) = \frac{n!z}{(z-1)^{n+1}} \quad for \quad |z| > 1.$$

Proof. We know that $Z((k+n-1)^{(n)}y_k) = (-1)^n z^n \frac{d^n}{dz^n} Y(z)$.

Let $y_k = 1$. Then

$$\begin{aligned}
Z((k+n-1)^{(n)}) &= (-1)^n z^n \frac{d^n}{dz^n} y(z) \\
&= (-1)^n z^n \frac{d^n}{dz^n} Z(y_k) \\
&= (-1)^n z^n \frac{d^n}{dz^n} Z(1) \\
&= (-1)^n z^n \frac{d^n}{dz^n} \Big(\frac{z}{z-1}\Big)
\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n z^n \frac{d^n}{dz^n}\frac{(-1)^n n!}{(z-1)^{n+1}} \\
&= \frac{n! z^n}{(z-1)^{n+1}}
\end{aligned}
$$

Now, we know that

$$
\begin{aligned}
Z(y_{k+n}) &= z^n Z(y_k) - \sum_{m=0}^{n-1} y_m z^{n-m}. \\
\Longrightarrow\ Z((k+n-1)^{(n)}) &= z^{n-1} Z(k^n) - \sum_{m=0}^{n-2} m^n z^{n-1-m} \\
\Longrightarrow\ \frac{n! z^n}{(z-1)^{n+1}} &= z^{n-1} Z(k^n) \\
&\quad - \sum_{m=0}^{n-2} m(m-1)\dots(m-n+2)(m-n+1) z^{n-1-m} \\
\Longrightarrow\ z^{n-1} Z(k^{(n)}) &= \frac{n! z^n}{(z-1)^{n+1}} \\
\Longrightarrow\ Z(k^{(n)}) &= \frac{n! z}{(z-1)^{n+1}} \quad for \quad |z| > 1.
\end{aligned}
$$

□

Theorem 3.2.18. ***(Initial and Final value Theorem)***

(a) If Y(z) exists for $|z| > r$, then $y_0 = \lim_{z \to \infty} Y(z)$.

(b) If Y(z) exists for $|z| > 1$ and (z-1)Y(z) is analytic at z=1, then
$\lim_{k \to \infty} y_k = \lim_{z \to 1} (z-1) Y(z)$.

Proof. (a) From the definition of z-transform,

$$
\begin{aligned}
Y(z) &= \sum_{k=0}^{\infty} \frac{y_k}{z^k} \\
&= y_0 + \frac{y_1}{z} + \frac{y_2}{z^2} + \cdots .
\end{aligned}
$$

Taking limits on both sides, $\lim_{z \to \infty} Y(z) = y_0$.

(b) Consider

$$
\begin{aligned}
Z(y_{k+1}) &= \sum_{k=0}^{\infty} y_{k+1} z^{-k} - \sum_{k=0}^{\infty} y_k z^{-k} \\
&= \lim_{n \to \infty} \Big[\sum_{k=0}^{n} y_{k+1} z^{-k} - \sum_{k=0}^{n} y_k z^{-k} \Big]
\end{aligned}
$$

$$\begin{aligned}
Z(y_{k+1}) &= \lim_{n\to\infty}\Big[(y_1 + y_2 z^{-1} + y_3 z^{-2} + \cdots + y_{n+1} z^{-n}) \\
&- (y_0 + y_1 z^{-1} + y_2 z^{-2} + \cdots + y_n z^{-n})\Big] \\
&= \lim_{n\to\infty}\Big[y_0 + y_1(1 - z^{-1}) + y_2(z^{-1} - z^{-2}) + \ldots \\
&\quad + y_n(z^{(-n-1)} - z^{-n}) + y_{n+1} z^{-n})\Big].
\end{aligned}$$

Thus

$$\lim_{z\to 1} Z(y_{k+1} - y_k) = \lim_{n\to\infty}[-y_0 + y_{n+1}].$$

From the Shifting theorem,

$$\begin{aligned}
\lim_{z\to 1}[zY(z) - zy_0 - Y(z)] &= \lim_{k\to\infty}[y_k - y_0] \\
\Longrightarrow \lim_{z\to 1}[(z-1)Y(z)] - y_0 &= \lim_{k\to\infty} y_k - y_0 \\
\Longrightarrow \lim_{z\to 1}[(z-1)Y(z)] &= \lim_{k\to\infty} y_k.
\end{aligned}$$

Hence the theorem. □

Example 3.2.19. *Verify the above theorem for the sequence* $y_k = 1$.

For (a),

$$1 = y_0 = \lim_{z\to\infty} Z(1)$$

and

$$\begin{aligned}
\lim_{z\to\infty} Y(z) &= \lim_{z\to\infty} Z(y_k) \\
&= \lim_{z\to\infty} Z(1) \\
&= \lim_{z\to\infty} \frac{z}{z-1} \\
&= \lim_{z\to\infty} \frac{z}{z(1-\frac{1}{z})} \\
&= \lim_{z\to\infty} \frac{1}{(1-\frac{1}{z})} \\
\lim_{z\to\infty} Y(z) &= 1.
\end{aligned}$$

Hence (a) is verified.

For (b),

$$
\begin{aligned}
\lim_{k\to\infty} y_k &= \lim_{k\to\infty} 1 = 1 \\
\textit{and} \quad \lim_{z\to 1}[(z-1)Y(z)] &= \lim_{z\to 1}[(z-1)Z(y_k)] \\
&= \lim_{z\to 1}[(z-1)Z(1)] \\
&= 0 + \lim_{z\to 1}(z-1)\frac{z}{z-1} \\
&= \lim_{z\to 1} z \\
&= 1.
\end{aligned}
$$

Hence (b) is verified.

Theorem 3.2.20. *If $Z(y_k) = Y(z)$ for $|z| > r$, then for constants $a \neq 0$, $Z(a^k y_k) = Y(\frac{z}{a})$ for $|z| > r|a|$.*

Proof. By the definition of z-transform,

$$
\begin{aligned}
Z(a^k y_k) &= \sum_{k=0}^{\infty} \frac{a^k y_k}{z^k} \\
&= \sum_{k=0}^{\infty} \frac{y_k}{(\frac{z}{a})^k} \\
&= Y(\frac{z}{a}).
\end{aligned}
$$

□

Example 3.2.21. *Find $Z(3^k sin4k)$.*

We know that,

$$
\begin{aligned}
Z(sin4k) &= \frac{zsin4}{z^2 - 2zcos4 + 1} \\
\implies Z(3^k sin4k) &= \frac{(z/3)sin4}{(z/3)^2 - 2(z/3)cos4 + 1} \\
&= \frac{(z/3)sin4}{(z^2/9) - 2(z/3)cos4 + 1} \\
\implies Z(3^k sin4k) &= \frac{3zsin4}{z^2 - 6zcos4 + 9}
\end{aligned}
$$

Example 3.2.22. *Solve the following initial value problem using z-transform*

$$
y_{k+1} - 3y_k = 4,
$$

$$
y_0 = 1.
$$

Taking the z-transform on both sides we get,

$$\begin{aligned}
Z(y_{k+1}) - 3Z(y_k) &= 4Z(1) \\
\implies zZ(y_k) - zy_0 - 3Z(y_k) &= 4\frac{z}{z-1} \\
\implies zY(z) - zy_0 - 3Y(z) &= \frac{4z}{z-1} \\
\implies zY(z) - z - 3Y(z) &= \frac{4z}{z-1} \quad here \quad y_0 = 1
\end{aligned}$$

$$\begin{aligned}
\implies Y(z)(z-3) &= z + \frac{4z}{z-1} \\
\implies Y(z)(z-3) &= \frac{z^2 - z + 4z}{z-1} \\
\implies Y(z) &= \frac{z^2+3z}{(z-1)(z-3)} \\
&= \frac{z(z+3)}{(z-1)(z-3)} \\
&= z\Big[\frac{-2}{z-1} + \frac{3}{z-3}\Big] \\
&= \frac{-2z}{z-1} + \frac{3z}{z-3}
\end{aligned}$$

$$\begin{aligned}
Y(z) &= -2Z(1) + 3Z(3^k) \\
\implies Z(y_k) &= -2Z(1) + 3Z(3^k) \\
\implies y_k &= -2 + 3 \cdot 3^k \\
\implies y_k &= -2 + 3^{k+1}.
\end{aligned}$$

Example 3.2.23. *Solve the initial value problem*

$$y_{k+1} - 3y_k = 3^k,$$

$$y_0 = 2.$$

Taking the z-transform on both sides we get,

$$\begin{aligned}
Z(y_{k+1}) - 3Z(y_k) &= Z(3^k) \\
\implies zY(z) - zy_0 - 3Y(z) &= \frac{z}{z-3} \\
\implies zY(z) - 2z - 3Y(z) &= \frac{z}{z-3}
\end{aligned}$$

$$
\begin{aligned}
\implies zY(z)(z-3) &= 2z + \frac{z}{z-3} \\
&= \frac{2z^2-6z+z}{(z-3)^2} \\
\implies Y(z) &= \frac{2z^2-5z}{(z-3)^2} \\
&= \frac{z(2z-5)}{(z-3)^2} \\
\implies Y(z) &= z\Big[\frac{2}{z-3} + \frac{1}{(z-3)^2}\Big] \\
\implies Y(z) &= \frac{2z}{z-3} + \frac{1}{3}\cdot\frac{3z}{(z-3)^2} \\
\implies Z(y_k) &= 2Z(3^k) + \frac{1}{3}Z(k3^k) \\
\implies y_k &= 2\cdot 3^k + \frac{1}{3}k3^k.
\end{aligned}
$$

Result: Use the binomial theorem to show that

$$
Z\Big(\binom{r}{k}\Big) = \Big(\tfrac{z+1}{z}\Big)^r, |z| > 1.
$$

$$
\begin{aligned}
Z\Big(\binom{r}{s}\Big) &= \sum_{k=0}^{\infty} \frac{\binom{r}{k}}{z^k} \\
&= \binom{r}{0} + \frac{\binom{r}{1}}{z} + \frac{\binom{r}{2}}{z^2} + \cdots + \frac{\binom{r}{r-1}}{z^{r-1}} + \frac{\binom{r}{r}}{z^r} \\
&= 1 + \frac{\binom{r}{1}}{z} + \frac{\binom{r}{2}}{z^2} + \cdots + \frac{\binom{r}{r-1}}{z^{r-1}} + \frac{1}{z^r} \\
&= 1 + \binom{r}{1}\frac{1}{z} + \binom{r}{2}\frac{1}{z^2} + \cdots + \binom{r}{r-1}\frac{1}{z^{r-1}} + \frac{1}{z^r} \\
&= 1 + r\cdot\frac{1}{z} + \frac{r(r-1)}{2!}\frac{1}{z^2} + \cdots + r\cdot\frac{1}{z^{r-1}} + \frac{1}{z^r}
\end{aligned}
\tag{3.12}
$$

$$
\begin{aligned}
&= \frac{1}{z^r}\Big[z^r + rz^{r-1} + \frac{r(r-1)}{2!}z^{r-2} + \cdots + rz + 1\Big] \\
&= \frac{(1+z)^r}{z^r} \\
Z\Big(\binom{r}{k}\Big) &= \Big(\frac{1+z}{z}\Big)^r.
\end{aligned}
$$

Example 3.2.24. *Solve the initial value problem*

$$(k+1)y_{k+1} - (50-k)y_k = 0,$$
$$y_0 = 1.$$

Taking the z-transform on both sides we get,

$$Z((k+1)y_{k+1}) - 50Z(y_k) + Z(ky_k) = 0. \tag{3.13}$$

Now, let us consider,

$$Z(y-k+1) = \sum_{k=0}^{\infty} \frac{y_{k+1}}{z^k}$$

$$\begin{aligned}
\implies zZ(y_k) - zy_0 &= \sum_{k=0}^{\infty} \frac{y_{k+1}}{z^k} \\
\implies z(Z(y_k) - y_0) &= \sum_{k=0}^{\infty} \frac{y_{k+1}}{z^k} \\
\implies Z(y_k) - 1 &= \sum_{k=0}^{\infty} \frac{y_{k+1}}{z^{k+1}}.
\end{aligned}$$

On differentiating, we get

$$\begin{aligned}
\frac{d}{dz}(Z(y_k) - 1) &= \frac{d}{dz}\sum_{k=0}^{\infty} \frac{y_{k+1}}{z^{k+1}} \\
\implies \frac{d}{dz}(Z(y_k) - 1) &= -\sum_{k=0}^{\infty} y_{k+1}(k+1)z^{-(k+2)} \\
\implies Y'(z) = &= -\sum_{k=0}^{\infty} y_{k+1}(k+1)z^{-k}z^{-2} \\
\implies z^2Y'(z) &= -\sum_{k=0}^{\infty} (k+1)\frac{y_{k+1}}{z^k} \\
\implies -z^2Y'(z) &= Z((k+1)y_{k+1}) \\
\implies z[-zY'(z)] &= Z((k+1)y_{k+1}) \\
\implies z[Z(ky_k)] &= Z((k+1)y_{k+1}).
\end{aligned} \tag{3.14}$$

Substituting (3.14) *in* (3.13), *we get*

$$\begin{aligned}
zZ(ky_k) - 50Z(y_k) + (-zY'(z)) &= 0 \\
\implies z(-zY'(z)) - 50Y(z) + (-zY'(z)) &= 0 \\
\implies -z^2Y'(z) - 50Y(z) - zY'(z) &= 0
\end{aligned}$$

$$
\begin{aligned}
\implies -(z^2+z)Y'(z) &= 50Y(z)\\
\implies -z(z+1)Y'(z) &= 50Y(z)\\
\implies \frac{Y'(z)}{Y(z)} &= \frac{-50}{z(z+1)}\\
&= -50\Big[\frac{1}{z}-\frac{1}{z+1}\Big]\\
\implies \frac{Y'(z)}{Y(z)} &= \frac{-50}{z}-\frac{50}{z+1}.
\end{aligned}
$$

On integrating, we get

$$
\begin{aligned}
logY(z) &= -50logz+50log(z+1)\\
\implies logY(z) &= log\Big(\frac{z+1}{z}\Big)^{50}\\
\implies Y(z) &= \Big(\frac{z+1}{z}\Big)^{50}\\
\implies Z(y_k) &= \Big(\frac{z+1}{z}\Big)^{50}.
\end{aligned}
$$

By using the above result,

$$
\begin{aligned}
Z(y_k) &= Z\Big(\begin{pmatrix} 50\\ k\end{pmatrix}\Big)\\
y_k &= \begin{pmatrix} 50\\ k\end{pmatrix}.
\end{aligned}
$$

Example 3.2.25. *Solve the second order initial value problem*

$$y_{k+2}+y_k = 10\cdot 3^k,$$

$$y_0=0, y_1=0.$$

Taking the z-transform on both sides we get,

$$
\begin{aligned}
Z(y_{k+2})+Z(y_k) &= 10Z(3^k)\\
\implies z^2Z(y_k)-y_0z^2-y_1z+Z(y_k) &= 10\frac{z}{z-3}\\
\implies (z^2+1)Z(y_k) &= 10\frac{z}{z-3}\\
\implies Z(y_k) &= 10\frac{z}{(z^2+1)(z-3)}\\
\implies Z(y_k) &= 10\Big[\frac{A}{z-3}+\frac{Bz+3}{z^2+1}\Big]\\
&= z\Big[\frac{1}{z-3}-\frac{z+3}{z^2+1}\Big]\\
&= \frac{z}{z-3}-\frac{z^2+3z}{z^2+1}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{z}{z-3} - \frac{z^2}{z^2+1} - \frac{3z}{z^2+1} \\
&= Z(3^k) - \frac{z^2 - zcos\pi/2}{z^2 - 2zcos\pi/2 + 1} - \frac{3zsin\pi/2}{z^2 - 2zcos\pi/2+1} \\
&= Z(3^k) - \frac{z(z - cos\pi/2)}{z^2 - 2zcos\pi/2 + 1} - \frac{3zsin\pi/2}{z^2 - 2zcos\pi/2+1} \\
\implies Z(y_k) &= Z(3^k) - Z(cos(\pi/2)k) - 3Z(cos(\pi/2)k) \\
\implies y_k &= 3^k - cos(\pi/2)k - 3cos(\pi/2)k.
\end{aligned}
$$

Example 3.2.26. *Solve the system,*

$$
u_{k+1} - v_k = 3k3^k \tag{3.15}
$$

$$
u_k + v_{k+1} - 3v_k = k3^k \tag{3.16}
$$

$$
u_0 = 0, v_0 = 3.
$$

Taking z-transform on both sides of equations (3.15) and (3.16), we have

$$
\begin{aligned}
Z(u_{k+1}) - Z(v_k) &= Z(k3^k) \\
\implies zU(z) - zu_0 - V(z) &= 3\frac{3z}{(z-3)^2} \\
\implies zU(z) - V(z) &= \frac{9z}{(z-3)^2} \qquad (3.17)
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \quad Z(u_k) + Z(v_{k+1}) - 3Z(v_k) &= Z(k3^k) \\
\implies U(z) + zV(z) - zv_0 - 3V(z) &= \frac{3z}{(z-3)^2} \\
\implies U(z) + zV(z) - 3z - 3V(z) &= \frac{3z}{(z-3)^2} \\
\implies U(z) + (z-3)V(z) - 3z &= \frac{3z}{(z-3)^2}. \qquad (3.18)
\end{aligned}
$$

Multiplying $z-3$ on both sides of equation (3.17), we get

$$
z(z-3)U(z) - (z-3)V(z) = \frac{9z}{z-3}. \tag{3.19}
$$

Adding (3.18) and (3.19), we get

$$
\begin{aligned}
U(z) + z(z-3)U(z) - 3z &= \frac{3z}{(z-3)^2} + \frac{9z}{(z-3)} \\
\implies (1 + z^2 - 3z)U(z) &= 3z + \frac{3z}{(z-3)^2} + \frac{9z}{(z-3)} \\
\implies (1 + z^2 - 3z)U(z) &= \frac{3z(z-3)^2 + 3z + 9z(z-3)}{(z-3)^2}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3z(z^2-6z+9)+3z+9z^2-27z}{(z-3)^2} \\
&= \frac{3z^3-18z^2+27z+3z+9z^2-27z}{(z-3)^2} \\
&= \frac{3z^3-9z^2+3z}{(z-3)^2} \\
\implies (z^2-3z+1)U(z) &= \frac{3z(z^2-3z+1)}{(z-3)^2} \\
\implies U(z) &= \frac{3z}{(z-3)^2} \\
\implies Z(u_k) &= Z(k3^k) \\
\implies u_k &= k3^k.
\end{aligned}
$$

Equation (3.15) becomes

$$
\begin{aligned}
(k+1)3^{k+1}-v_k &= 3k3^k \\
-v_k &= 3k3^k-(k+1)3k+1 \\
-v_k &= 3k3^{k+1}-k3k+1-3^{k+1} \\
-v_k &= -3^{k+1} \\
\implies v_k &= 3^{k+1}.
\end{aligned}
$$

Example 3.2.27. *Solve the system,*

$$u_{k+1}-v_k = -1 \tag{3.20}$$

$$-u_k+v_{k+1} = 3 \tag{3.21}$$

$$u_0=0, v_0=2$$

Taking z-transform on both sides of equations (3.15) and (3.16),

$$
\begin{aligned}
Z(u_{k+1})-Z(v_k) &= -Z(1) \\
zU(z)-zu_0-V(z) &= -\frac{z}{z-1} \\
zU(z)-V(z) &= \frac{z}{z-1} \qquad (3.22) \\
-Z(u_k)+Z(v_{k+1}) &= 3Z(1) \\
-U(z)+zV(z)-zv_0 &= \frac{3z}{z-1} \\
-U(z)+zV(z)-2z &= \frac{3z}{z-1} \\
-U(z)+(z-3)V(z) &= \frac{3z}{z-1}+2z \qquad (3.23)
\end{aligned}
$$

Multiplying z on both sides by equation (3.22), we get

$$z^2U(z) - zV(z) = \frac{3z^2}{z-1} \tag{3.24}$$

Adding (3.22) and (3.23), we get

$$\begin{aligned}
(z^2-1)U(z) &= \frac{3z}{z-1} - \frac{z^2}{z-1} + z \\
\implies (z^2-1)U(z) &= \frac{3z - z^2 + 2z^2 - 2z}{z-1} \\
\implies (z^2-1)U(z) &= \frac{z^2+z}{z-1} \\
\implies U(z) &= \frac{z(z+1)}{(z-1)(z^2-1)} \\
\implies U(z) &= \frac{z}{(z-1)^2} \\
\implies Z(u_k) = Z(k) \\
\implies u_k = k.
\end{aligned}$$

Equation (3.20) becomes

$$\begin{aligned}
(k+1) - v_k &= -1 \\
\implies -v_k &= -1 - k - 1 \\
\implies v_k &= k + 2.
\end{aligned}$$

Definition 3.2.28. *Unit impulse sequence $\delta(n), n \geq 1$ is defined by*

$$\delta_k(n) = \begin{cases} 1 & \textit{if} \quad k = n \\ 0 & \textit{if} \quad k \neq n. \end{cases}$$

Taking z-transform, we get

$$Z(\delta_k(n)) = \sum_{k=0}^{\infty} \frac{\delta_k(n)}{z^k} = \frac{1}{z^n}.$$

Example 3.2.29. *Solve the initial value problem*

$$y_{k+1} - 2y_k = 3\delta_k(4),$$

$$y_0 = 1.$$

Taking z-transform on both sides, we get

$$
\begin{aligned}
Z(y_{k+1}) - 2Z(y_k) &= 3Z(\delta_k(4)) \\
\implies zY(z) - zy_0 - 2Y(z) &= 3\frac{1}{z^4} \\
\implies zY(z) - z - 2Y(z) &= \frac{3}{z^4} \\
\implies (z-2)Y(z) &= \frac{3}{z^4} + z \\
\implies Y(z) &= \frac{3}{z^4(z-2)} + \frac{z}{z-2} \\
&= \frac{3z}{z^5(z-2)} + \frac{z}{z-2} \\
\implies Y(z) &= 3z^{-5}\frac{z}{(z-2)} + \frac{z}{z-2} \\
\implies Z(y_k) &= Z(2^k) + 3z^{-5}Z(2^k) \\
\implies Z(y_k) &= Z(2^k) + 3Z(2^{k-5}u_k(5)) \\
\implies y_k &= 2^k + 3 \cdot 2^{k-5}u_k(5)
\end{aligned}
$$

We do write y_k as

$$
y_k = \begin{cases} 2^k & \textit{if} \quad 0 \le k \le 4 \\ 2^k + 3 \cdot 2^{k-5} & \textit{if} \quad k \ge 5 \end{cases}
$$

Definition 3.2.30. *The convolution of two sequences $\{u_k\}$ and $\{v_k\}$ is defined by*

$$
\{u_k\} * \{v_k\} = \left\{ \sum_{m=0}^{k} u_{k-m}v_m \right\}
$$

We write $u_k * v_k = \sum_{m=0}^{k} u_{k-m}v_m$.

Theorem 3.2.31. ***Convolution Theorem***

If U(z) exists for $|z| > a$ and V(z) exists for $|z| > b$, then

$$
Z(u_k * v_k) = U(z)V(z)
$$

for $|z| > max\{a, b\}$.

Proof. For $|z| > max\{a, b\}$,

$$
\begin{aligned}
U(z)V(z) &= \sum_{k=0}^{\infty} \frac{u_k}{z^k} \sum_{k=0}^{\infty} \frac{v_k}{z^k} \\
&= \left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \dots\right]\left[v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \dots\right]
\end{aligned}
$$

$$
\begin{aligned}
&= u_0v_0 + (u_0v_1 + u_1v_0)\frac{1}{z} + (u_0v_2 + u_1v_1 + u_2v_0)\frac{1}{z^2} + \dots \\
&\quad +(u_0v_k + u_1v_{k-1} + u_2v_{k-2} + \dots u_kv_0)\frac{1}{z^k} + \dots \\
&= \sum_{k=0}^{\infty}\sum_{m=0}^{k}\frac{u_{k-m}v_m}{z^k} \\
&= \sum_{k=0}^{\infty}\frac{u_k * v_k}{z^k} \\
\implies U(z)V(z) &= Z(u_k * v_k).
\end{aligned}
$$

□

Corollary 3.2.32. *If $Z(y_k)$ exists for $|z| > r$, then $Z(\sum_{m=0}^{k} y_m) = \frac{z}{z-1}Z(y_k)$ for $|z| > max\{1, r\}$*

Proof. Take $u_k = 1$ in $\sum_{m=0}^{k} u_{k-m}v_m = u_k * v_k$.
Then $\sum_{m=0}^{k} 1y_m = 1 * y_k$ i.e., $\sum_{m=0}^{k} y_m = 1 * y_k$.

Taking z-transform on both sides, we get

$$
\begin{aligned}
Z(\sum_{m=0}^{k} y_m) &= Z(1 * y_k) \\
&= 1(z)Y(z) \\
&= Z(1)Z(y_k) \\
&= \frac{z}{z-1}Z(y_k).
\end{aligned}
$$

□

Example 3.2.33. *Find $Z(\sum_{m=0}^{k} 3^m)$*

By the above corollary,

$$
\begin{aligned}
Z(\sum_{m=0}^{k} 3^m) &= \frac{z}{z-1}Z(3^k) \\
&= \frac{z}{z-1}\frac{z}{z-3} \\
&= \frac{z^2}{(z-1)(z-3)}, \qquad (|z| > 3)
\end{aligned}
$$

The Volterra Summation Equation of convolution type

Consider the Volterra Summation Equation of Convolution type

$$
y_k = f_k + \sum_{m=0}^{k-1} u_{k-m-1}y_m, \qquad (k \geq 0) \tag{3.25}
$$

where f_k and u_{k-m-1} are given. The term u_{k-m-1} is called the kernal of the summation equation. The equation is said to be homogeneous if $f_k \equiv 0$ and non-homogeneous otherwise. Such an equation can often be solved by use of the z-transform.

To see this, replace k by k+1 in (3.25). Then

$$\begin{aligned} y_{k+1} &= f_{k+1} + \sum_{m=0}^{k} u_{k-m} y_m \\ \implies y_{k+1} &= f_{k+1} + u_k * y_k \end{aligned}$$

Taking the z-transform on both sides and put $y_0 = f_0$, we get

$$\begin{aligned} Z(y_{k+1}) &= Z(f_{k+1}) + Z(u_k * y_k) \\ \implies zY(z) - zy_0 &= zF(z) - zf_0 + U(z)Y(z) \\ \implies zY(z) - zf_0 &= zF(z) - zf_0 + U(z)Y(z) \\ \implies zY(z) &= zF(z) + U(z)Y(z) \\ \implies zY(z) - U(z)Y(z) &= zF(z) \\ \implies (z - U(z))Y(z) &= zF(z) \\ \implies Y(z) &= \frac{zF(z)}{z - U(z)}. \end{aligned}$$

Example 3.2.34. *Solve the Volterra summation equation*

$$y_k = 1 + 16 \sum_{m=0}^{k-1} (k - m - 1) y_m, \qquad k \geq 0. \tag{3.26}$$

Replacing k by k+1,

$$\begin{aligned} y_{k+1} &= 1 + 16 \sum_{m=0}^{k} (k - m) y_m \\ \implies y_{k+1} &= 1 + 16k * y_k. \end{aligned}$$

Taking the z-transform on both sides, we get

$$\begin{aligned} Z(y_{k+1}) &= Z(1 + 16(k * y_k)) \\ \implies Z(y_{k+1}) &= Z(1) + 16Z(k * y_k)) \\ \implies zY(z) - zy_0 &= \frac{z}{z-1} + 16Z(k)Z(y_k) \end{aligned}$$

$$
\begin{aligned}
\implies zY(z) - zy_0 &= \frac{z}{z-1} + 16\frac{z}{(z-1)^2}Z(y_k) \\
\implies Y(z) - y_0 &= \frac{1}{z-1} + 16\frac{1}{(z-1)^2}Z(y_k) \\
\implies (1 - \frac{16}{(z-1)^2})Y(z) &= 1 + \frac{z}{z-1} \\
\implies \Big[\frac{z^2 - 2z + 1 - 16}{(z-1)^2}\Big]Y(z) &= \frac{z-1+1}{z-1} \\
\implies \Big[\frac{z^2 - 2z - 15}{(z-1)^2}\Big]Y(z) &= \frac{z}{z-1} \\
\implies (z^2 - 2z - 15)Y(z) &= z(z-1) \\
\implies Y(z) &= \frac{z(z-1)}{z^2 - 2z - 15} \\
&= \frac{z(z-1)}{(z-5)(z+3)} \\
&= z\Big[\frac{\frac{1}{2}}{(z-5)} + \frac{\frac{1}{2}}{(z+3)}\Big] \\
\implies Y(z) &= \frac{z}{2(z-5)} + \frac{z}{2(z+3)} \\
\implies Z(y_k) &= \frac{1}{2}Z(5^k) + \frac{1}{2}Z((-3)^k) \\
\implies y_k &= \frac{1}{2}5^k + \frac{1}{2}(-3)^k
\end{aligned}
$$

The Fredholm Summation Equation

Theorem 3.2.35. *The Fredholm equation*

$$y_k = f_k + \sum_{m=a}^{b} K_{k,m} y_m, \qquad (a \le k, m \le b) \tag{3.27}$$

where a and b are integers and f_k is sequence, with a separable kernal

$$K_{k,m} = \sum_{i=1}^{p} \alpha_i(k)\beta_i(m), \qquad (a \le k, m \le b)$$

has a solution y_k iff

$$(I - A)\overline{c} = \overline{u} \tag{3.28}$$

has a solution $\overline{c}$. If $\overline{c} = [c_1, c_2, \ldots, c_p]^T$ is a solution of equation (3.28), then a corresponding solution y_k of equation (3.27) is given by

$$y_k = f_k + \sum_{i=1}^{p} c_i \alpha_i(k), \qquad (a \le k \le b)$$

where I is a $p \times p$ identity matrix.

Proof. Substitute the expression for $K_{k,m}$ in equation (3.27), we get

$$\begin{aligned} y_k &= f_k + \sum_{m=a}^{b}\sum_{i=1}^{p} \alpha_i(k)\beta_i(m)y_m \\ \implies y_k &= f_k + \sum_{i=1}^{p} \alpha_i(k)(\sum_{m=a}^{b} \beta_i(m)y_m) \\ y_k &= f_k + \sum_{i=1}^{p} \alpha_i(k)c_i, \end{aligned} \tag{3.29}$$

where $c_i = \sum_{m=a}^{b} \beta_i(m)y_m$.

Multiplying by $\beta_j(k)$ on both sides of (3.29) and summing from a to b, we get

$$\begin{aligned} \sum_{k=a}^{b} y_k\beta_j(k) &= \sum_{k=a}^{b} \beta_j(k)f_k + \sum_{i=1}^{p} c_i \sum_{k=a}^{b} \alpha_i(k)\beta_j(k) \\ \implies c_j &= u_j + \sum_{i=1}^{p} a_{ji}c_i, 1 \leq j \leq p \end{aligned} \tag{3.30}$$

where

$$\begin{aligned} c_j &= \textstyle\sum_{k=a}^{b} y_k\beta_j(k), \\ u_j &= \textstyle\sum_{k=a}^{b} \beta_j(k)f_k, \\ a_{ji} &= \textstyle\sum_{k=a}^{b} \alpha_i(k)\beta_j(k). \end{aligned}$$

Let A be the $p \times p$ matrix, $A = (a_{ij})$.

Let $\overline{c} = [c_1, c_2, \ldots, c_p]^T$ and $\overline{u} = [u_1, u_2, \ldots, u_p]^T$.

Then

$$\overline{u} + A\overline{c} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix} + [a_{ij}] \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \vdots & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pp} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}$$

$$= \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_p \end{pmatrix} + \begin{pmatrix} \sum_{i=1}^{p} a_{1i}c_i \\ \sum_{i=1}^{p} a_{2i}c_i \\ \vdots \\ \sum_{i=1}^{p} a_{pi}c_i \end{pmatrix}$$

$$= \begin{pmatrix} u_1 + \sum_{i=1}^{p} a_{1i}c_i \\ u_2 + \sum_{i=1}^{p} a_{2i}c_i \\ \vdots \\ u_p + \sum_{i=1}^{p} a_{pi}c_i \end{pmatrix}$$

$$= \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{pmatrix}$$

(by equation (3.30)).

$$\therefore \overline{u} + A\overline{c} = \begin{pmatrix} c_1 & c_2 & \ldots & c_p \end{pmatrix}^T$$

$\implies \overline{u} + A\overline{c} = \overline{c}$

$\implies \overline{u} = \overline{c} - A\overline{c}$

$\implies \overline{u} = (I - A)\overline{c}$

Thus $(I - A)\overline{c} = \overline{u}$, where I is the $p \times p$ identity matrix. □

Example 3.2.36. *Solve the Fredholm summation equation*

$$y_k = 1 + \sum_{m=0}^{19}(1 + km)y_m, \qquad 0 \leq k \leq 19.$$

Comparing the given equation with equation (3.27)*, we get* $\delta_k = 1, K_{k,m} = 1 + km,$ $a = 0, b = 19.$

$\implies \alpha_1(k) = 1, \beta_1(m) = 1, \alpha_2(k) = k, \beta_2(m) = m.$

We know that,

$$\begin{aligned}
a_{ij} &= \sum_{k=a}^{b} \alpha_j(k)\beta_i(k). \\
\Longrightarrow & \\
a_{11} &= \sum_{k=a}^{19} 1\cdot 1 = 20, \\
a_{12} &= \sum_{k=a}^{19} k\cdot 1 = \frac{19\times(19+1)}{2} = 190, \\
a_{21} &= \sum_{k=a}^{19} k\cdot 1 = \frac{19\times(19+1)}{2} = 190, \\
a_{22} &= \sum_{k=a}^{19} k\cdot k = \frac{19\times(19+1)\times(2\times 19+1)}{6} = \frac{19\times 20\times 39}{6} = 2470.
\end{aligned}$$

Now,

$$\begin{aligned}
u_j &= \sum_{k=a}^{b} f_k\beta_j(k) \\
\Longrightarrow & \\
u_1 &= \sum_{k=0}^{19} 1\cdot 1 = 20, \\
u_2 &= \sum_{k=0}^{19} 1\cdot k = 190.
\end{aligned}$$

$$\therefore A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 20 & 190 \\ 190 & 2470 \end{pmatrix}$$

,

$$\overline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 190 \end{pmatrix}$$

Substituting A and $\overline{u}$ in equation (3.28)*, we have*

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 20 & 190 \\ 190 & 2470 \end{pmatrix}\right]\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 190 \end{pmatrix}$$

$$\begin{pmatrix} -19 & -190 \\ -190 & -2469 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 20 \\ 190 \end{pmatrix}$$

$$-19c_1 - 190c_2 = 20 \qquad (3.31)$$

$$-190c_1 - 2469c_2 = 190 \qquad (3.32)$$

$10 \times ((3.31)) - ((3.32)) \implies 569c_2 = 10 \implies c_2 = \frac{10}{569}$.

Substituting c_2 *in* (3.31), *we get*

$-19c_1 = \frac{13280}{569} \implies c_1 = -\frac{13280}{10811}$. *Therefore,*

$$\begin{aligned} y_k &= f_k + \sum_{i=1}^{p} c_i \alpha_i(k) \\ &= c_1\alpha_1(k) + c_2\alpha_2(k) \\ &= 1 - \frac{13280}{10811} + \frac{10}{569}k \\ &= \frac{-2469}{10811} + \frac{10}{569}k, \qquad 0 \leq k \leq 19. \end{aligned}$$

Example 3.2.37. *Solve the Fredholm summation equation*

$$y_k = 2 + \lambda \sum_{m=0}^{29} \frac{m}{29} y_m, \qquad 0 \leq k \leq 29,$$

for all values of λ.

Comparing the given equation with equation (3.27),

we get $f_k = 2, K_{k,m} = \frac{m}{29}\lambda, a = 0, b = 29$ *which implies* $\alpha_1(k) = \lambda$ *and* $\beta_1(m) = \frac{m}{29}$.

From

$$\begin{aligned} a_{ij} &= \sum_{k=a}^{b} \alpha_j(k)\beta_i(k), \\ a_{11} &= \sum_{m=a}^{29} \frac{\lambda m}{29} = \frac{\lambda}{29}\sum_{k=a}^{29} m \\ &= \frac{\lambda}{29} \times \frac{29 \times 30}{2} \\ &= 15\lambda \end{aligned}$$

and from

$$\begin{aligned} u_j &= \sum_{k=a}^{b} f_k \beta_j(k), \\ u_1 &= \sum_{m=0}^{29} 1\frac{2m}{29} \\ &= \frac{2}{29} \times \frac{29 \times 30}{2} \\ &= 30. \end{aligned}$$

Substituting all the values in $(I-A)\overline{c}=\overline{u}$, we get

$$\begin{aligned}(1-15\lambda)c &= 30\\ \implies c &= \frac{30}{1-15\lambda}, \qquad \lambda\neq\frac{1}{15}\\ y_k &= 2+\frac{30}{1-15\lambda}\\ \therefore y_k &= \frac{2}{1-15\lambda}\end{aligned}$$

Remark 3.2.38. *Consider the homogeneous Fredholm equation*

$$y_k=\lambda\sum_{m=a}^{b}K_{k,m}y_m, \quad a\leq k\leq b \tag{3.33}$$

where λ is a parameter. We say that λ_0 is an eigenvalue of this equation, provided that for this value of λ, there is a nontrivial solution y_k, called an eigensequence. We say that (λ_0, y_k) is an eigenpair for Eq.(3.33). Note that $\lambda=0$ is not an eigenvalue. We say that $K_{k,m}$ is symmetric provided that

$$K_{k,m}=K_{m,k}$$

for $a\leq k,m\leq b$. Several properties of eigenpairs for Eq.(3.33) with a symmetric kernel are given in the following theorem.

Theorem 3.2.39. *If $K_{k,m}$ is real and symmetric, then all the eigenvalues of Eq. (3.33) are real. If $(\lambda_i,u_k)\,(\lambda_j,v_k)$ are eigenpairs with $\lambda_i\neq\lambda_j$, then u_k and v_k are orthogonal; that is,*

$$\sum_{k=a}^{b}u_kv_k=0$$

We can always pick a real eigensequence that corresponds to each eigenvalue.

Proof. Let $(\mu,u_k)\,,(\nu,v_k)$ be eigenpairs of Eq. (3.33). Then $\mu,v\neq 0$. Since $(\ \mu,u_k)$ is an eigenpair for Eq. (3.33),

$$u_k=\mu\sum_{m=a}^{b}K_{k,m}u_m$$

Multiplying by v_k and summing from a to b, we obtain

$$\begin{aligned}\sum_{k=a}^{b} u_k v_k &= \mu \sum_{k=a}^{b} \sum_{m=a}^{b} K_{k,m} u_m v_k \\ &= \mu \sum_{m=a}^{b} \left(\sum_{k=a}^{b} K_{m,k} v_k \right) u_m \\ &= \frac{\mu}{v} \sum_{m=a}^{b} v_m u_m\end{aligned}$$

since (v, v_k) is an eigenpair for Eq. (3.33). It follows that

$$(v - \mu) \sum_{k=a}^{b} u_k v_k = 0 \tag{3.34}$$

If $\mu \neq \nu$, we get the orthogonality result

$$\sum_{k=a}^{b} u_k v_k = 0$$

If (λ_i, y_k) is an eigenpair of Eq. (3.33), then $\left(\bar{\lambda}_i, \bar{y}_k\right)$ is an eigenpair of Eq. (3.33). With $(\mu, u_k) = (\lambda_i, y_k)$ and $(\nu, v_k) = \left(\bar{\lambda}_i, \bar{y}_k\right)$, Eq. (3.34) becomes

$$(\bar{\lambda} - \lambda) \sum_{k=a}^{b} y_k \bar{y}_k = 0$$

It follows that $\lambda = \bar{\lambda}$, and hence every eigenvalue of Eq. (3.33) is real. □

z-transforms of some functions

Sequence	Z-transform
1	$\frac{z}{z-1}$
a^k	$\frac{z}{z-a}$
k	$\frac{z}{(z-1)^2}$
k^2	$\frac{z(z+1)}{(z-1)^3}$
k^n	$\frac{n!(z+1)}{(z-1)^{n+1}}$
$sinak$	$\frac{zsina}{z^2-2zcosa+1-1}$
$cosak$	$\frac{z^2-zcosa}{z^2-2zcosa+1}$
$sinhak$	$\frac{zsinha}{z^2-2zcosha+1}$
$coshak$	$\frac{z^2-zcosha}{z^2-2zcosha+1}$
$\delta_k(n)$	$\frac{1}{z^n}$
$u_k(n)$	$\frac{z^{n-1}}{(z-1)}$
$u_k * v_k$	$U(z)V(z)$
$\sum_{m=0}^{k} y_i$	$\frac{z}{(z-1)}Y(z)$
$a^k y_k$	$Y\left(\frac{z}{a}\right)$

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. z-transform of a sequence
2. Exponentially bounded sequence
3. Linearity Theorem
4. Shifting theorem
5. Initial value and final value theorem
6. Solving initial value problems using z-transforms
7. Convolution Theorem
8. Solving the Volterra summation equation
9. Solving the Fredholm summation equation

Check your Progress:

1. The $z-$ transform of the sequence $\{y_k = 1\}$ is ....

   (A) $\frac{z-1}{z}$ (B) $\frac{z}{z-1}$ (C) $\frac{z}{z+1}$ (D) None of these

2. If the sequence $\{y_k\}$ is exponentially bounded , then the z-transform of $\{y_k\}$

   (A) converges (B) does not exists (C) is unbounded (D) diverges

3. $Z(ka^k) =$.............

   (A) $\frac{z}{z-a}$ (B) $\frac{az}{z-a}$ (C) $\frac{az}{(z-a)^2}$ (D) None of these

Unit Summary:

In this unit, several methods are given for solving certain difference equations with variable coefficients and some summation equations.

Glossary:

- $Z(y_k)$ or $Y(z)$ - The z-transform of a sequence $\{y_k\}$
- $\delta(n)$ -The unit impulse sequence
- $\{u_k\} * \{v_k\}$ -The convolution of the sequences $\{u_k\}$ and $\{v_k\}$

Self-Assessment Questions:

1. Find the general solution of

$$(E - (t+1))(E+1)u(t) = 0.$$

2. Check that $u_n = 2^n$ solves

$$nu_{n+2} - (1+2n)u_{n+1} + 2u_n = 0$$

   and find a second independent solution.

3. Find the z-transform of each of the following:

   (a) $y_k = 2 + 3k$.

   (b) $u_k = 3^k \cos 2k$.

   (c) $v_k = \sin(2k-3)$.

   (d) $y_k = k^3$.

   (e) $u_k = 3y_{k+3}$.

   (f) $v_k = k \cos \frac{k\pi}{2}$.

   (g) $y_k = \frac{1}{k!}$.

   (h) $u_k = \begin{cases} \frac{(-1)^{\frac{k}{2}}}{(k+1)!} & k \text{ even} \\ 0, & k \text{ odd.} \end{cases}$

Exercises:

1. Solve the equation $y(t+2) + (2t-1)y(t+1) - 6ty(t) = 0$ by factoring.

2. Use the method of reduction of order to solve the difference equation $u_{n+2} - 5u_{n+1} + 6u_n = 0$, given that $u_n = 3^n$ is a solution.

3. Use the method of generating functions to solve

$$3(n+2)u_{n+2} - (3n+4)u_{n+1} + u_n = 0$$

   if $u_0 = 3u_1$.

4. Solve the first order initial value problem $y_{k+1} - 3y_k = 4^k, \quad y_0 = 0$ using z-transforms.

5. Solve the second order initial value problem $y_{k+2} - 5y_{k+1} + 6y_k = 0, y_0 = 1, y_1 = 0$ using z-transforms.

6. Solve the system using z-transforms:

$$\begin{aligned} u_{k+1} - 2v_k &= 2 \cdot 4^k \\ -4u_k + v_{k+1} &= 4^{k+1} \\ u_0 = 2, \quad v_0 &= 3. \end{aligned}$$

7. Solve the Fredholm summation equation: $y_k = 10 + \sum_{m=0}^{20} kmy_m$.

Answers for check your progress:

Section 3.1	1. (B)	2. (C)	3. (B)
Section 3.2	1. (B)	2. (A)	3. (C)

References:

1. W.G. Kelley and A.C. Peterson, "Difference Equations", 2nd Edition, Academic Press, New York, 2001.

Suggested Reading:

1. R.P. Agarwal, "Difference Equations and Inequalities", 2nd Edition, Marcel Dekker, New York, 2000.

2. S.N. Elaydi, "An Introduction to Difference Equations", 3rd Edition, Springer, India, 2008.

3. R. E. Mickens, "Difference Equations", 3rd Edition, CRC Press, 2015.

UNIT 4

Unit 4

Stability Theory

Objectives:

This unit deals with initial value problems and stability theory for homogeneous linear systems.

4.1 Initial Value Problems for Linear Systems

Consider systems of the form

$$\begin{aligned} u_1(t+1) &= a_{11}(t)u_1(t) + \cdots + a_{1n}(t)u_n(t) + f_1(t) \\ u_2(t+1) &= a_{21}(t)u_1(t) + \cdots + a_{2n}(t)u_n(t) + f_2(t) \\ &\vdots \\ u_n(t+1) &= a_{n1}(t)u_1(t) + \cdots + a_{nn}(t)u_n(t) + f_n(t) \end{aligned}$$

for $t = a, a+1, a+2, \cdots$. This system can be written as an equivalent vector equation,

$$u(t+1) = A(t)u(t) + f(t), \tag{4.1}$$

where $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$, $f(t) = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$ and

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}.$$

The study of (4.1) includes the n^{th} order scalar equation

$$p_n(t)y(t+n) + \cdots + p_0(t)y(t) = r(t) \tag{4.2}$$

as a special case. To see this, let $y(t)$ solve (4.2) and define

$$u_i(t) = y(t+i-1)$$

for $1 \le i \le n, t = a, a+1, \ldots$. That is,

$$\begin{aligned} u_1(t) &= y(t) \\ u_2(t) &= y(t+1) \\ &\vdots \\ u_{n-1}(t) &= y(t+n-2) \\ u_n(t) &= y(t+n-1), \end{aligned}$$

and $u_i(t+1) = y(t+i)$, and so

$$\begin{aligned} u_1(t+1) &= y(t+1) = u_2(t) \\ u_2(t+1) &= y(t+2) = u_3(t) \\ &\vdots \\ u_{n-1}(t+1) &= y(t+n-1) = u_n(t) \\ u_n(t+1) &= y(t+n). \end{aligned}$$

From (4.2), we have

$$\begin{aligned} y(t+n) &= -\frac{p_0(t)}{p_n(t)}y(t) - \frac{p_1(t)}{p_n(t)}y(t+1) - \cdots - \frac{p_{n-1}(t)}{p_n(t)}y(t+n-1) + \frac{r(t)}{p_n(t)} \\ &= -\frac{p_0(t)}{p_n(t)}u_1(t) - \frac{p_1(t)}{p_n(t)}u_2(t) - \cdots \frac{p_{n-1}(t)}{p_n(t)}u_n(t) + \frac{r(t)}{p_n(t)} \end{aligned}$$

$$\begin{bmatrix} u_1(t+1) \\ u_2(t+1) \\ \vdots \\ u_n(t+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{p_0(t)}{p_n(t)} & -\frac{p_1(t)}{p_n(t)} & -\frac{p_2(t)}{p_n(t)} & \cdots & \frac{-p_{n-1}(t)}{p_n(t)} \end{bmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{r(t)}{p_n(t)} \end{bmatrix}$$

Then the vector function $u(t)$ with components $u_1(t)$ satisfies equation (4.1) if

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -\frac{p_0(t)}{p_n(t)} & \frac{p_1(t)}{p_n(t)} & -\frac{p_2(t)}{p_n(t)} & \cdots & \frac{-p_{n-1}(t)}{p_n(t)} \end{bmatrix} \text{ and } \quad f(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \frac{r(t)}{p_n(t)} \end{bmatrix} \tag{4.3}$$

The matrix $A(t)$ in the above equation is called “Companian Matrix" of equation (4.2). Conversely, if $u(t)$ solves (4.1) with $A(t)$ and $f(t)$ given in Eq. (4.3), then $y(t) = u_1(t)$ is a solution of (4.2). So, we have the following theorem.

Theorem 4.1.1. *For each t_0 in $\{a, a+1, \cdots\}$ and each n-vector u_0, (4.1) has a unique solution $u(t)$ defined for $t = t_0, t_0+1, \cdots$, so that $u(t_0) = u_0$.*

Remark: Assume that A is independent of t (i.e., all coefficients in the system are constants) and $f(t) = 0$. Then the solution $u(t)$ of

$$u(t+1) = Au(t) \tag{4.4}$$

satisfying the initial condition $u(0) = u_0$, is $u(t) = A^t u_0, (t = 0, 1, 2, \cdots)$. Hence the solutions of (4.4) can be found by calculating powers of A.

Definition 4.1.2. *The equation*

$$Au = \lambda u \tag{4.5}$$

where λ is a parameter, always has the trivial solution $u = 0$. If (4.5) has a nontrivial solution u for some λ, then λ is called an eigenvalue of A and u is called a corresponding eigenvector of A.

Note: The eigenvalues of A satisfy the characteristic equation $\det(\lambda I - A) = 0$, where I is the n by n identity matrix.

Definition 4.1.3. *An eigenvalue is said to be simple if its multiplicity as a root of the characteristic equation is one.*

Definition 4.1.4. *The spectrum of A, denoted $\sigma(A)$, is the set of eigenvalues of A, and the spectral radius of A is*

$$r(A) = \max\{|\lambda| : \lambda \textit{ is in } \sigma(A)\}.$$

Example 4.1.5. *Find the eigenvalues, eigenvectors, and spectral radius for*

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

The characteristic equation of A is

$$\begin{aligned} \det(\lambda I - A) &= 0 \\ \implies \det\left[\lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right] &= 0 \\ \implies \begin{vmatrix} \lambda & \lambda - 1 \\ 2 & \lambda + 3 \end{vmatrix} &= 0 \\ \implies \lambda^2 + 3\lambda + 2 &= 0 \end{aligned}$$

$$\therefore \lambda = -2, \lambda = -1.$$

So, the eigen values are given by $\sigma(A) = \{-2, -1\}$.

To find the eigen vectors corresponding to $\lambda = -2$*, we solve* $(\lambda I - A)u = 0$.

$$\Longrightarrow \left[-2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The eigen vectors are all non-zero multiples of the vector with $u_1 = 1$ *and* $u_2 = -2$.

To find eigen vector corresponding to $\lambda = -1$*; we solve* $(\lambda I - A)u = 0$.

$$\Longrightarrow \left[-1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} -u_1 - u_2 \\ 2u_1 + 2u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The eigen vectors corresponding to $\lambda = -1$ *are all non-zero multiples of the vector with* $u_1 = 1$ *and* $u_2 = -1$*. Finally the spectral radius of* A *is*

$$r(A) = \max\{|-2|, |-1|\}.$$

$$\Longrightarrow r(A) = 2.$$

Result: Let λ be an eigenvalue of A and let u be a corresponding eigenvector. For $t = 0, 1, 2, \cdots$, we have

$$A^t u = \lambda^t u$$

so $u(t) = \lambda^t u$ satisfies (4.4) with initial vector u. Also if u_0 can be written as a linear combination of the eigenvectors of A, say

$$u_0 = b_1 u^1 + \cdots + b_k u^k$$

where each u^i is an eigenvector corresponding to λ_i, then the solution of (4.4) is

$$u(t) = b_1 \lambda_1^t u^1 + \cdots + b_k \lambda_k^t u^k. \tag{4.6}$$

Thus, if A has n linearly independent eigenvectors (this is necessarily the case if A has n distinct eigenvalues or if A is symmetric), then every solution of the system can be calculated in above said way.

Example 4.1.6. *Solve* $u(t+1) = Au(t)$ *if* $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Let $u_0 = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ *be an initial vector and recall that* $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ *is an eigenvector for* $\lambda = -2$

and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ *is an eigenvector for* $\lambda = -1$. *Now, set*

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + b_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}.$$

The solution of this linear system is

$$\begin{aligned} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} -u_1 - u_2 \\ 2u_1 + u_2 \end{bmatrix} \end{aligned}$$

By (4.6), *the solution of* (4.4) *with initial vector* u_0 *is*

$$u(t) = -(u_1 + u_2)(-2)^t \begin{bmatrix} 1 \\ -2 \end{bmatrix} + (2u_1 + u_2)(-1)^t \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Result: The Cayley-Hamilton Theorem:

Every square matrix satisfies its characteristic equation.

Example 4.1.7. *Verify the Cayley-Hamilton Theorem for*

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

The characteristic equation for A is

$$\det \begin{bmatrix} \lambda - 1 & -2 \\ -3 & \lambda - 4 \end{bmatrix} = \lambda^2 - 5\lambda - 2 = 0.$$

Now

$$\begin{aligned} A^2 - 5A - 2I &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 5 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+6 & 2+8 \\ 8+12 & 6+16 \end{bmatrix} - \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

i.e., A satisfies its characteristic equation.

$\therefore$ Cayley Hamilton theorem is verified.

Remark 4.1.8. *Let A be $n \times n$ matrix. If $\lambda_1, \cdots, \lambda_n$ are eigen values of A, then Cayley-Hamilton Theorem implies that A^n can be written as a linear combination of $I, A, A^2, \cdots, A^{n-1}$.Thus, every power of A also can be written as a linear combination of $I, A, A^2, \cdots, A^{n-1}$.*

Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A, with each eigenvalue repeated as many times as its multiplicity. Define

$$M_0 = I$$
$$M_i = (A - \lambda_i I)\, M_{i-1}, \quad (1 \le i \le n). \tag{4.7}$$

Then

$$\begin{aligned}
M_1 &= (A - \lambda_1 I)\, M_0 \\
&= (A - \lambda_1 I)\, I \\
&= A - \lambda_1 I \\
M_2 &= (A - \lambda_2 I)\, M_1 \\
&= (A - \lambda_2 I)\,(A - \lambda_1 I) \\
\textit{Similarly}, M_3 &= (A - \lambda_3 I)\,(A - \lambda_2 I)\,(A - \lambda_1 I) \\
&\vdots \\
M_n &= (A - \lambda_n I)\,(A - \lambda_{n-1} I)\,(A - \lambda_{n-2} I) \cdots (A - \lambda_1 I)\,.
\end{aligned}$$

It follows from the Cayley-Hamilton Theorem that $M_n = 0$.

Theorem 4.1.9. *The solution of $u(t+1) = Au(t)$, where A is independent of t with initial vector u_0 is*

$$u(t) = \sum_{i=0}^{n-1} c_{i+1}(t) M_i u_0 = A^t u_0,$$

where the $M_i's$ are given by $M_0 = I, M_i = (A - \lambda_i I) M_{i-1}, (1 \le i \le n)$ and the $c_i(t), (i = 1, \ldots, n)$ are uniquely determined by

$$\begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_n(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{4.8}$$

and

$$\begin{bmatrix} c_1(t+1) \\ \vdots \\ c_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{bmatrix}. \tag{4.9}$$

Proof. We know that the solution of $u(t+1) - Au(t) = 0$ (with the initial condition $u(0) = u_0$) is given by

$$\begin{aligned} u(t) &= u(0) \prod_{s=0}^{t-1} P(s) \\ &= u(0) \prod_{s=0}^{t-1} A \\ &= u_0 A^t. \end{aligned}$$

Let us find A^t.

Now, by the definition of M_i, each A^i is a linear combination of $M_0, \dots, M_i$ for $i = 0, \dots, n-1$, and by the remark above, the same is true for every power of A. Then, we can write

$$A^t = \sum_{i=0}^{n-1} c_{i+1}(t) M_i$$

for $t \geq 0$.

$$\begin{aligned} \therefore u(t) &= u_0 A^t \\ &= u_0 \sum_{i=0}^{n-1} c_{i+1}(t) M_i \\ &= \sum_{i=0}^{n-1} c_{i+1}(t) M_i u_0, t \geq 0 \end{aligned}$$

where the $c_{i+1}(t)$ are to be determined. Since $A^{t+1} = A \cdot A^t$, we have

$$\begin{aligned} \sum_{i=0}^{n-1} c_{i+1}(t+1) M_i &= A \sum_{i=0}^{n-1} c_{i+1}(t) M_i \\ &= \sum_{i=0}^{n-1} c_{i+1}(t) A M_i \\ &= \sum_{i=0}^{n-1} c_{i+1}(t) \left[M_{i+1} + \lambda_{i+1} M_i \right] \\ &= \sum_{i=1}^{n-1} c_i(t) M_i + \sum_{i=0}^{n-1} c_{i+1}(t) \lambda_{i+1} M_i \ , \end{aligned}$$

where we have replaced i by $i-1$ in the first sum and used the fact that $M_n = 0$.

The preceding equation is satisfied if the $c_i(t), (i = 1, \ldots, n)$ are chosen to satisfy the system,

$$\begin{bmatrix} c_1(t+1) \\ \vdots \\ c_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(t) \\ \vdots \\ c_n(t) \end{bmatrix}. \tag{4.10}$$

Since

$$A^0 = I \quad = \quad c_1(0)I + c_2(0)M_1 + \cdots + c_n(0)M_{n-1},$$

we must have

$$\begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_n(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.11}$$

By Theorem 4.1.1, the initial value problem (4.10), (4.11) has a unique solution. Hence the theorem. □

Example 4.1.10. *Solve*

$$u(t+1) = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} u(t), \quad u(0) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}.$$

Comparing the given equation with $u(t+1) = Au(t)$, we get $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$.

The characteristic equation is,

$$|\lambda I - A| = 0, \qquad \text{where } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

$$\Longrightarrow \left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \right| = 0$$

$$\Longrightarrow \begin{vmatrix} \lambda - 1 & -1 \\ 1 & \lambda - 3 \end{vmatrix} = 0$$

$$\Longrightarrow \lambda^2 - 3\lambda - \lambda + 3 + 1 = 0$$

$$\Longrightarrow \lambda^2 - 4\lambda + 4 = 0 \Rightarrow \lambda = 2, 2$$

The matrix A has an eigen value $\lambda = 2$ of multiplicity 2.

By equation (4.7), $M_0 = I, M_1 = A - 2I$

$$\therefore M_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, M_1 = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$$

From equations (4.10) and (4.11),

$$\begin{bmatrix} c_1 & (t+1) \\ c_2 & (t+1) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 & (t) \\ c_2 & (t) \end{bmatrix} \text{ and } \begin{bmatrix} c_1(0) \\ c_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

$$\implies c_1(t+1) = 2c_1(t), \quad c_1(0) = 1,$$

$$c_2(t+1) = c_1(t) + 2c_2(t), \quad c_2(0) = 0.$$

First, consider $c_1(t+1) = 2c_1(t), \quad c_1(0) = 1.$

$$\begin{aligned} \implies c_1(t+1) &= 2c_1(t) = 0 \\ \implies c_1(t) &= c_1(0)\prod_{s=0}^{t-1} 2 \\ \implies c_1(t) &= 1 \cdot 2^t \\ \implies c_1(t) &= 2^t. \end{aligned}$$

Next, consider

$$c_2(t+1) = 2c_2(t) + c_1(t), \quad c_2(0) = 0.$$

$$\therefore c_2(t+1) = 2c_2(t) + 2^t.$$

Then

$$c_2(t) = t \cdot 2^{t-1}.$$

By Theorem (4.1.9), we have

$$\begin{aligned} u(t) &= (c_1(t)M_0 + c_2(t)M_1)\, u_0 \\ &= (c_1(t)I + c_2(t)M_1) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= \left(2^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t2^{t-1} \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\right) \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ &= 2^t \begin{bmatrix} 1 - \frac{t}{2} & \frac{t}{2} \\ -\frac{t}{2} & 1 + \frac{t}{2} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}. \end{aligned}$$

Definition 4.1.11. *A matrix 'A' is said to be nilpotent if* $A^{t-1} \neq 0$ *and* $A^t = 0$ *where* $t \in \mathbb{Z}^+$.

Example 4.1.12. *Compute all powers of*

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}.$$

Write

$$A^t = \left(2I + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)^t$$

Now, since $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ *is nilpotent and commutes with* I*, the binomial theorem yields,*

$$\begin{aligned} A^t &= 2^t I^t + t2^{t-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= 2^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t2^{t-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^t & 0 \\ 0 & 2^t \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ t2^{t-1} & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2^t & 0 \\ t2^{t-1} & 2^t \end{bmatrix}. \end{aligned}$$

Finally, consider the nonhomogeneous system

$$u(t+1) = Au(t) + f(t) \tag{4.12}$$

The next theorem is a variation of parameters formula for solving (4.12).

Theorem 4.1.13. *The solution of* (4.12) *satisfying the initial condition* $u(0) = u_0$ *is*

$$u(t) = A^t u_0 + \sum_{s=0}^{t-1} A^{t-s-1} f(s). \tag{4.13}$$

Proof. It is sufficient to show that (4.13) satisfies the initial value problem (4.12).

From equation (4.13),

$$u(0) = A^0 u_0 + \sum_{s=0}^{-1} A^{-s-1} f(s).$$

First we have

$$\sum_{s=0}^{-1} A^{-s-1} f(s) = 0$$

by the usual convention, so $u(0) = u_0$.

For $t \geq 1$,

$$\begin{aligned} u(t+1) &= A^{t+1} u_0 + \sum_{s=0}^{t} A^{t-s} f(s) \\ &= A^{t+1} u_0 + \sum_{s=0}^{t-1} A^{t-s} f(s) + f(t) \\ &= A \left[A^t u_0 + \sum_{s=0}^{t-1} A^{t-s-1} f(s) \right] + f(t) \\ &= Au(t) + f(t) \end{aligned}$$

∴ The equation (4.13) is the solution of equation (4.12). □

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Solving initial value problems for linear systems
2. Companion matrix
3. The Cayley-Hamilton Theorem
4. The Putzer algorithm
5. Computing the powers of a square matrix

Check your Progress:

1. The statement "every square matrix satisfies its own characteristic equations" is known as ....

   (A) Convolution theorem (B) Cayley-Hamilton Theorem

   (C Putzer algorithm (D) None of these

2. The spectrum of A is denoted by ....

   (A) $\sigma(A)$ (B) $r(A)$ (C) $s(A)$ (D) None of these

3. If $Au = \lambda u$ has a non-trivial solution u for some λ, then λ is called ....

   (A) spectral radius of A (B) eigenvector of A

   (C eigenvalue of A (D) None of these

4.2 Stability of Linear Systems

The solution of an initial value problem for a system of equations with n unknowns is represented geometrically by a sequence of points $\{(u_1(t), u_2(t), \ldots, u_n(t))\}_{t=0}^{\infty}$ in $\mathbb{R}^n$. In many of the applications of this subject,it is useful to know the general location of those points for large values of t. Of course, there are numerous possibilities. the sequence could converge to a point at at least remain near a point, the sequence could oscillate among values near several points, the sequence might become unbounded or the sequence might remain in a bounded set but jump around in seemingly unpredictable fashion. The study of these matters is called stability theory.

Theorem 4.2.1. *Let A be an n by n matrix with $r(A) < 1$. Then every solution $u(t)$ of (4.4) satisfies $\lim_{t\to\infty} u(t) = 0$. Furthermore, if $r(A) < \delta < 1$, then there is a constant $C > 0$ so that*

$$|u(t)| \leq C\delta^t |u(0)| \tag{4.14}$$

for $t \geq 0$ and every solution of u of (4.4).

Proof. Fix δ so that $r(A) < \delta < 1$, the solution of equation $u(t+1) = Au(t), u(0) = u_0$ is

$$u(t) = \sum_{i=0}^{n-1} c_{i+1}(t) M_i u_0,$$

where $c_i(t)$ are given by

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \\ \vdots \\ c_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{bmatrix}$$

and $\begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_n(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$.

By equation (4.10),

$$c_1(t+1) = \lambda_1 c_1(t).$$

$$\begin{aligned} \implies \mid c_1(t+1) \mid &= |\lambda_1 c_1(t)| = |\lambda_1| \, |c_1(t)| \\ &\leq r(A) \, |c_1(t)| \, . \end{aligned}$$

Iterating this inequality and using $c_1(0) = 1$, we have

$$\begin{aligned} &\mid c_1(1)) \mid \leq r(A), \\ &|c_1(2)| \leq r(A) \, |c_1\,(1)| \Rightarrow |c_1(2)| \leq (r(A))^2, \\ &|c_1\,(3))| \leq r(A) \, |c_1(2)| \Rightarrow |c_1\,(3))| \leq (r(A))^3, \\ &\vdots \\ &|c_1(t)| \leq (r(A))^t \leq \delta^t, t \geq 0. \end{aligned}$$

Again,

$$c_2(t+1) = c_1(t) + \lambda_2 c_2(t).$$

$$\begin{aligned} \implies |c_2(t+1)| &\leq |c_1(t)| + |\lambda_2 c_2(t)| \\ &\leq |c_1(t)| + r(A)\,|c_2(t)| \\ \implies |c_2(t+1)| &\leq r(A)\,|c_2(t)| + (r(A))^t. \end{aligned}$$

It follows from iteration and $c_2(0) = 0$ that

$$\begin{aligned} |c_2(1)| &\leqslant 1, \\ |c_2\,(2)| &\leq (r(A))^1 + r(A)\,|c_2\,(2)| = 2r(A) \\ \implies |c_2(2)| &\leqslant 2r(A), \\ |c_2(3)| &\leqslant (r(A))^2 + r(A) \mid c_2(2) \mid = (r(A))^2 + 2(r(A))^2 \\ \implies |c_2(3)| &\leqslant 3(r(A))^2, \\ &\vdots \\ |c_2(t)| &\leqslant t(r(A))^{t-1}, t \geq 0 \\ \implies |c_2(t)| &\leqslant t\left(\frac{r(A)}{\delta}\right)^{t-1} \delta^{t-1}. \end{aligned}$$

Now,

$$\begin{aligned} \lim_{t\to\infty}\left(t\left(\frac{r(A)}{8}\right)^{t-1}\right) &= \frac{\delta}{r(A)} \lim_{t\to\infty} t\left(\frac{r(A)}{\delta}\right)^t \\ &= \frac{\delta}{r(A)} \lim_{t\to\infty} \frac{t}{\left(\frac{r(A)}{\delta}\right)^{-t}} \\ &= \frac{\delta}{r(A)} \lim_{t\to\infty} \frac{1}{\frac{-\log\left(\frac{r(A)}{\delta}\right)}{\left(\frac{r(A)}{\delta}\right)^t}}. \end{aligned}$$

$$\implies \lim_{t\to\infty}\left(t\left(\frac{r(A)}{8}\right)^{t-1}\right) = \frac{-\delta}{r(A)} \frac{1}{\log\left(\frac{r(A)}{\delta}\right)} \lim_{t\to\infty}\left(\frac{r(A)}{\delta}\right)^t \to 0 \text{ as } t \to \infty$$

$$\therefore t\left(\frac{r(A)}{\delta}\right)^{t-1} \text{ converges to } 0 \text{ as } t \to \infty$$

$$\implies \quad t\left(\frac{r(A)}{\delta}\right)^{t-1} \text{ is a bounded sequence.}$$

So, there is a constant $B_1 > 0$ such that

$$| c_2(t) | \leqslant B_1 \delta^t, t \geq 0.$$

Similarly, we can show that for $t \geq 0$,

$$|c_3(t)| \leqslant \frac{t(t-1)}{2}(r(A))^{t-2} \text{ and}$$

there is a constant $B_2 > 0$ so that

$$|c_3(t)| \leqslant B_2 \delta^t, t \geq 0.$$

Continuing in this way (by induction), we obtain a constant $B^* > 0$ so that

$$|c_i(t)| \leqslant B^* \delta^t, \quad t \geq 0, \quad i = 1, 2, \ldots n.$$

Now, for any matrix M, there is a constant $D > 0$ so that

$$|Mv| \leq D|v| \quad \forall v \text{ in } \mathbb{R}^n.$$

Finally, the solution $u(t)$ of equation

$$u(t+1) = Au(t), \quad u(0) = u_0 \quad \text{satisfies}$$

$$\begin{aligned}
|u(t)| &= \left| \sum_{i=0}^{n-1} c_{i+1}(t) M_i u_0 \right| \\
&\leq \sum_{i=0}^{n-1} |c_{i+1}(t)| \, |M_i u_0| \\
&\leq B^* \delta^t \, |u_0| \sum_{i=0}^{n-1} D_i \\
&= C\delta^t \, |u_0|
\end{aligned}$$

$$\implies |u(t)| \leq C\delta^t \, |u_0| \, , \text{for } C = B^* \sum_{i=0}^{n-1} D_i.$$

Consequently, $|u(t)| \leqslant C\delta^t \, |u_0|$ holds. Since $0 < \delta < 1$, $\lim_{t \to \infty} u(t) = 0$. □

Note: When all the solutions of the system go to the origin as t goes to infinity, the origin is said to be **"asymptotically stable"**.

Theorem 4.2.2. *If* $r(A) \geq 1$, *some solution* $u(t)$ *of* (4.4) *does not go to the origin as* t *goes to infinity.*

Proof. Since $r(A) \geq 1$, there is an eigenvalue λ of A so that $|\lambda| \geq 1$. Let v be a corresponding eigenvector. Then $u(t) = \lambda^t v$ is a solution of (4.4) and $|u(t)| = |\lambda|^t |v| \nrightarrow 0$ as $t \to \infty$. $\square$

Example 4.2.3. *Solve* $u(t+1) = \begin{bmatrix} 1 & -5 \\ .25 & -1 \end{bmatrix} u(t)$.

The characteristic equation of $A = \begin{bmatrix} 1 & -5 \\ .25 & -1 \end{bmatrix}$ *is given by*

$$\begin{aligned} |A - \lambda I| &= 0 \\ \implies \begin{vmatrix} \lambda - 1 & -5 \\ -25 & \lambda + 1 \end{vmatrix} &= 0 \\ \implies \lambda^2 - 1 + \frac{5}{4} &= 0 \\ \implies \lambda^2 &= -1/4 \\ \implies \lambda &= \pm i/2. \end{aligned}$$

Then, $\sigma(A) = \{i/2, -i/2\}$ *and*

$$r(A) = \max\{|i/2|, |-i/2|\} \quad \implies r(A) = 1/2 < 1.$$

$\therefore$ *All solutions of this system converge to the origin as* $t \to \infty$.

Theorem 4.2.4. *Assume that*

(a) $r(A) \leq 1$.

(b) Each eigenvalue λ *of* A *with* $|\lambda| = 1$ *is simple.*

Then there is a constant $C > 0$ *such that*

$$|u(t)| \leq C\,|u_0| \tag{4.15}$$

for $t \geq 0$ *and every solution* u *of* (4.4).

Proof. Label the eigen values of A so that $|\lambda_i| = 1$ for $i = 1, 2, \ldots, k-1$ and $|\lambda_i| < 1$ for $i = k, \ldots n$.

From theorem (4.1.9), we have

$$\begin{bmatrix} c_1(t+1) \\ c_2(t+1) \\ \vdots \\ c_n(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 1 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 1 & \lambda_3 & \cdots & 0 \\ 0 & \cdots & 1 & \cdots & \lambda_n \end{bmatrix} \begin{bmatrix} c_1(t) \\ c_2(t) \\ \vdots \\ c_n(t) \end{bmatrix}$$

and $\begin{bmatrix} c_1(0) \\ c_2(0) \\ \vdots \\ c_n(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$

Now, consider

$$\begin{aligned} c_1(t+1) &= \lambda_1 c_1(t), \quad c_1(0) = 1. \\ c_1(t+1) - \lambda_1 c_1(t) &= 0 \\ \implies c_1(t) &= c_1(0) \prod_{s=0}^{t-1} \lambda_1 \\ &= (\lambda_1 \lambda_1 \ldots \lambda_1) \\ \implies c_1(t) &= \lambda_1^t. \end{aligned}$$

Next, consider

$$\begin{aligned} c_2(t+1) &= c_1(t) + \lambda_2 c_2(t), \quad c_2(0) = 0 \\ \implies c_2(t+1) &= \lambda_1^t + \lambda_2 c_2(t), \\ \therefore c_2(t+1) - \lambda_2 c_2(t) &= \lambda_1^t. \end{aligned}$$

Since $c_2(0) = 0$, we solve this equation by using annihilator method.

$$\begin{aligned} c_2(t+1) - \lambda_2 c_2(t) &= \lambda_1^t \\ \implies (E - \lambda_2)\, c_2(t) &= \lambda_1^t. \end{aligned}$$

Since ${\lambda_1}^t$ satisfies the homogeneous equation

$$(E - \lambda_1)\, \lambda_1^t = 0,$$

we have

$$(E - \lambda_1)\,(E - \lambda_2)\, c_2(t) = 0.$$

Since $\lambda_1 \neq \lambda_2$, the general solution is

$$c_2(t) = B_{12}\lambda_1^t + B_{22}\lambda_2^t \text{ for some constants } B_{12}, B_{22}.$$

Continuing in this way, we have

$$c_i(t) = B_{1i}\lambda_1^t + B_{2i}\lambda_2^t + \ldots + B_{ii}\lambda_i^t, \text{ for } i = 1, \ldots, k-1$$

Consequently, there is a constant $D > 0$ so that

$$
\begin{aligned}
|c_i(t)| &= \left|B_{1i}\lambda_1^t + B_{2i}\lambda_2^t + \ldots + B_{ii}\lambda_i^t\right| \\
&\leq \left|B_{1i}\lambda_1^t\right| + \left|B_{2i}\lambda_2^t\right| + \ldots + \left|\ B_{ii}\lambda_i t\right| \\
&= |B_{1i}| + |B_{2i}| + \cdots + |B_{ii}|\,. \\
\implies\ |c_i(t)| &\leq D.
\end{aligned}
$$

From (4.10),

$$
\begin{aligned}
c_k(t+1) &= c_{k-1}(t) + \lambda_k c_k(t), \quad c_k(0) = 0 \\
\implies |c_k(t+1)| &= |c_{k-1}(t) + \lambda_k c_k(t)| \\
&\leq |c_{k-1}(t)| + |\lambda_k|\,|c_k(t)| \\
&\leq D + |\lambda_k|\,|c_k(t)|\,.
\end{aligned}
$$

Choose $\delta = \max\{|\lambda_k|\,, |\lambda_{k+1}|\,, \ldots |\lambda_{n-1}|\,, |\lambda_n|\} < 1$

Then

$$|c_k(t+1)| \leq D + \delta\,|c_k(t)|\,.$$

Since $c_k(0) = 0$, by iteration

$$
\begin{aligned}
|c_k(t)| &\leq D\sum_{j=0}^{t-1}\delta^j \\
&= D\left[1 + \delta + \delta^2 + \cdots + \delta^{t-1}\right] \\
&\leq D[1-\delta]^{-1} \\
&= \frac{D}{1-\delta}
\end{aligned}
$$

That is, $\quad |c_k(t)| \leq \dfrac{D}{1-\delta},$ for $t \geq 0$.

In a similar manner, we find that there is a constant D^* so that $|c_i(t)| \leq D^*$ for $i = 1, \ldots n \ and \ \ t \geq 0$.

Thus, the solution of equation $u(t+1) = Au(t)$, $u(0) = u_0$ is given by

$$
\begin{aligned}
u(t) &= \sum_{i=0}^{n-1} c_{i+1}(t) M_i u_0. \\
\implies |u(t)| &\leq \sum_{i=0}^{n-1} |c_{i+1}(t)|\, |M_i u_0| \\
&\leq D^* \sum_{i=0}^{n-1} |M_i u_0| \\
&= D^* \left[|M_0 u_0| + |M_1 u_0| + \cdots + |M_{n-1} u_0| \right] \\
&\leq D^* \left[D_0 |u_0| + D_1 |u_0| + \cdots + D_{n-1} |u_0| \right] \\
&= D^* \left(D_0 + D_1 + \cdots + D_{n-1} \right) |u_0| \\
&= D^* D |u_0| \text{ , where } D = D_0 + D_1 + \cdots + D_{n-1} \\
&= C|u_0|, \text{where } C = D^* D > 0 \text{ is a constant.}
\end{aligned}
$$

$$
\therefore |u(t)| \leq C|u_0| \quad \text{for } t \geq 0 \text{ and some } C > 0.
$$

□

Definition 4.2.5. *Let A be an $n \times n$ matrix. Let λ be an eigenvalue of A of multiplicity m. Then, the generalized eigenvectors of A corresponding to λ are the non-trivial solutions v of $(A - \lambda I)^m v = 0$. Clearly, every eigenvector of A is also a generalized eigenvector. The set of all generalized eigenvectors corresponding to λ, together with the zero vector, is called the generalized eigenspace and is a vector space having dimension m.*

Note:

1) The intersection of any two generalized eigenspaces is the zero vector.

2) A times a generalized eigenvector is a vector in the same generalized eigenspace.

Example 4.2.6. *What are the generalized eigenvectors for*

$$
A = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} ?
$$

The characteristic equation of A is $|\lambda I - A| = 0$.

$$
\implies \begin{vmatrix} \lambda - 3 & 1 & 0 \\ 0 & \lambda - 3 & 0 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = 0
$$

$$
\implies (\lambda - 3)(\lambda - 3)(\lambda - 2) = 0
$$

$$
\implies (\lambda - 3)^2(\lambda - 2) = 0
$$

Therefore, A has eigen values $\lambda_1 = 3$ of multiplicity two and $\lambda_2 = 2$.

Then, the generalized eigen vector corresponding to $\lambda_1 = 3$ are solutions of $(A-3I)^2 v = 0$.

$$\implies \left[\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right]^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies v_3 = 0.$$

Thus, the generalized eigenspace consists of all vectors with $v_3 = 0$. This is a two dimensional space, and

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

are basis vectors.

Next, the generalized eigen vector corresponding to $\lambda_2 = 2$ are solutions of $(A-2I)v = 0$.

$$\implies \left[\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right] \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\implies v_1 + v_2 = 0, \quad v_2 = 0.$$

Therefore, the generalized eigenspace consists of all vectors with $v_1 = 0$ and $v_2 = 0$. This is a one-dimensional (generalized) eigenspace spanned by the eigenvector

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Theorem 4.2.7. ***(The Stable Subspace Theorem)*** *Let $\lambda_1, \cdots, \lambda_n$ be the (not necessarily distinct) eigenvalues of A arranged so that $\lambda_1, \cdots, \lambda_k$ are the eigenvalues with $|\lambda_i| < 1$. Let S be the k-dimensional space spanned by the generalized eigenvectors corresponding to $\lambda_1, \cdots, \lambda_k$. If u is a solution of Eq. (4.4) with $u(0)$ in S, then $u(t)$ is in S for $t \geq 0$ and*

$$\lim_{t \to \infty} u(t) = 0.$$

Proof. Let u be a solution of Eq. (4.4) with $u(0)$ in S. Since A takes every generalized eigenspace into itself, it also takes S into itself. Then $u(t)$ is in S for $t \geq 0$. Choose δ

so that

$$\max\{|\lambda_1|, \cdots, |\lambda_k|\} < \delta < 1.$$

As in the proof of Theorem (4.2.1) , there is a constant $B > 0$ such that

$$|c_i(t)| \leq B\delta^t$$

for $t \geq 0, 1 \leq i \leq k$. By Theorem (4.1.9)

$$u(t) = \sum_{i=0}^{n-1} c_{i+1}(t)M_i u(0).$$

Recalling the definition of M_i, Eq. (4.7) and the fact that $u(0)$ is a linear combination of generalized eigenvectors corresponding to $\lambda_1, \cdots, \lambda_k$, we have, for $i \geq k$,

$$M_i u(0) = 0.$$

Then

$$\begin{aligned} |u(t)| &\leq \sum_{i=0}^{k-1} |c_{i+1}(t)| \, |M_i u(0)| \\ &\leq B\delta^t \sum_{i=0}^{k-1} |M_i u(0)| \\ &\leq C\delta^t |u(0)|, \quad (t \geq 0) \end{aligned}$$

for some constant C,

so

$$\lim_{t\to\infty} u(t) = 0.$$

□

Note: The set S in Theorem (4.2.7) is called the "stable subspace" for Eq. (4.4). It can be shown that every solution of the system that goes to the origin as t tends to infinity must have its initial point in S. Thus, S can be described as the union of all sequences $\{u(t)\}_{t=0}^{\infty}$ that solve the system and satisfy $\lim_{t\to\infty} u(t) = 0$.

Example 4.2.8. *What is the stable subspace for the system*

$$u(t+1) = \begin{bmatrix} .5 & 0 & 0 \\ 1 & .5 & 0 \\ 0 & 1 & 2 \end{bmatrix} u(t)?$$

The characteristic equation is

$$\det\begin{bmatrix} \lambda-.5 & 0 & 0 \\ -1 & \lambda-.5 & 0 \\ 0 & -1 & \lambda-2 \end{bmatrix} = (\lambda-.5)^2(\lambda-2)=0.$$

Then, A has eigen values $\lambda_1 = .5$ of multiplicity two and $\lambda_2 = 2$.

Since $|\lambda_1| < 1$, he stable subspace has dimension two and consists of the solutions of

$$(A-.5I)^2 v = 0.$$

$$\Longrightarrow \left[\begin{bmatrix} .5 & 0 & 0 \\ 1 & .5 & 0 \\ 0 & 1 & 2 \end{bmatrix} - \begin{bmatrix} .5 & 0 & 0 \\ 0 & .5 & 0 \\ 0 & 0 & .5 \end{bmatrix}\right]^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \frac{-3}{2} \end{bmatrix}^2 \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & \frac{3}{2} & \frac{9}{4} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Longrightarrow v_1 + \frac{3}{2}v_2 + \frac{9}{4}v_3 = 0$$

$$\Longrightarrow 4v_1 + 6v_2 + 9v_3 = 0.$$

Thus, S is the plane

$$4v_1 + 6v_2 + 9v_3 = 0.$$

From Theorem (4.2.7), every solution that originates in this plane remains in the plane for all values of t and converges to the origin as $t \to \infty$. Since $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is an eigenvector corresponding to $\lambda = 2$, the solutions originating on the v_3 axis are given by

$$u(t) = 2^t \begin{bmatrix} 0 \\ 0 \\ v_3 \end{bmatrix}, \quad (t \geq 0)$$

These remain on the v_3 axis and approach infinity in the positive or negative direction, depending on whether v_3 is positive or negative.

Remark *If some of the eigenvalues λ of A with $|\lambda| < 1$ are complex numbers, then the corresponding generalized eigenvectors will also be complex, and the stable subspace is a complex vector space. However, those generalized eigenvectors occur in conjugate pairs, and it is not difficult to verify that the real and imaginary parts of these vectors are real vectors that generate a real stable subspace of the same dimension.*

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Asymptotically stable
2. Generalized eigenvectors
3. The Stable Subspace Theorem
4. Finding the stable subspace for the given system

Check your Progress:

1. If A is an $n \times n$ matrix with $r(A) < 1$, then every solution $u(t)$ of $u(t+1) = Au(t)$ satisfies ....

   (A) $\lim_{t\to\infty} u'(t) = 0$ (B) $\lim_{t\to\infty} u'(t) \neq 0$

   (C $\lim_{t\to\infty} u(t) = 0$ (D) $\lim_{t\to\infty} u(t) \neq 0$

2. When all solutions of the system go to the origin as $t \to \infty$, the origin is said to be ....

   (A) asymptotically stable (B) stable (C) exponentially stable (D) unstable

3. If A is an $n \times n$ matrix, and λ is an eigenvalue of A of multiplicity m, then the generalized eigenvectors of A corresponding to λ are the non-trivial solutions v of ....

   (A) $(A - \lambda I)^n v = 0$ (B) $(A - \lambda I)^m v = 0$

   (C $(A - \lambda I)^{n-1} v = 0$ (D) $(A - \lambda I)^{m-1} v = 0$

Unit Summary:

In this unit, the solution of the initial value problem for homogeneous linear systems is derived through Putzer algorithm. Further, how the concept of generalized eigen vectors is used to find the stable subspace of the given system is explained.

Glossary:

- $\det(\lambda I - A) = 0$ - Characteristic equation of the matrix A

- $\sigma(A)$ - The spectrum of A

- $r(A)$ - The spectral radius of A

Self-Assessment Questions:

1. Convert the following second order system

$$v(t+2) - 6v(t+1) + 4w(t+1) - 3v(t) + w(t) = 0$$

$$w(t+2) + w(t+1) + 3v(t+1) - 2w(t) = t3^t$$

   into a first order system like Eq.(4.1)

2. Show that the characteristic equation for

$$y(t+2) + ay(t+1) + by(t) = 0$$

   is the same as the characteristic equation of the companion matrix.

3. Verify the Cayley-Hamilton Theorem for $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

4. For which of the following systems do all solutions converge to the origin as $t \to \infty$?

   (a) $u(t+1) = \begin{bmatrix} .9 & .2 \\ -.1 & .6 \end{bmatrix} u(t)$. (b) $u(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -3 & 3 \end{bmatrix} u(t)$.

5. Find the stable subspace S for the following:
   $u(t+1) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix} u(t)$.

Exercises:

1. Use Eq. (4.6) to solve $u(t+1) = Au(t)$ if

   (a) A is the matrix $\begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$.

   (b) A is the matrix $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

2. Use Theorem (4.1.9) to solve

$$y(t+2) + 3y(t+1) + 2y(t) = 0$$

$$y(0) = -1, \quad y(1) = 7$$

3. Find $A^t, (t \geq 0)$ for the following matrix A. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

4. Solve, using Theorem (4.1.13)
$u(t+1) = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad u(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

5. Prove the following: if the characteristic equation for

$$y(t+n) + p_{n-1}y(t+n-1) + \cdots + p_0 y(t) = 0$$

has a multiple characteristic root λ with $|\lambda| = 1$, then the difference equation has an unbounded solution.

6. Find the real two-dimensional stable subspace for

$$u(t+1) = \begin{bmatrix} 0 & \frac{1}{2} & -1 \\ \frac{3}{2} & 1 & 0 \\ 0 & -\frac{5}{6} & 1 \end{bmatrix} u(t)$$

Answers for Check your Progress:

Section 4.1	1. (B)	2. (A)	3. (C)
Section 4.2	1. (C)	2. (A)	3. (B)

References:

1. W.G. Kelley and A.C. Peterson, "Difference Equations", 2nd Edition, Academic Press, New York, 2001.

Suggested Reading:

1. R.P. Agarwal, "Difference Equations and Inequalities", 2nd Edition, Marcel Dekker, New York, 2000.

2. S.N. Elaydi, "An Introduction to Difference Equations", 3rd Edition, Springer, India, 2008.

3. R. E. Mickens, "Difference Equations", 3rd Edition, CRC Press, 2015.

UNIT 5

Unit 5

Asymptotic Methods

Objectives:

This unit deals with approximations of solutions of difference equations for large values of the independent variable.

5.1 Introduction

This section does not deal with difference equations but does present a number of basic concepts and tools of asymptotic analysis.

Definition 5.1.1. *If* $\lim_{t\to\infty} \frac{y(t)}{z(t)} = 1$*, then we say that "*$y(t)$ *is asymptotic to* $z(t)$ *as* t *tends to infinity" and write*

$$y(t) \sim z(t), \quad (t \to \infty).$$

Example 5.1.2.

$$\begin{aligned}
\lim_{t\to\infty} \frac{(4t^2+4)^{3/2}}{8t^3} &= \lim_{t\to\infty} \frac{(4t^2)^{3/2}(1+1/4t)^{3/2}}{8t^3} \\
&= \lim_{t\to\infty} \frac{(2t)^3\left(1+\frac{1}{4t}\right)^{\frac{3}{2}}}{8t^3} \\
&= \lim_{t\to\infty} \left(1+\frac{1}{4t}\right)^{3/2} \\
&= 1 \\
\therefore \left(4t^2+t\right)^{3/2} &\sim 8t^3, \quad (t\to\infty).
\end{aligned}$$

Example 5.1.3.

$$\lim_{t\to\infty}\frac{\left(\frac{1}{3t^2+2t}\right)}{\frac{1}{3t^2}} = \lim_{t\to\infty}\frac{1}{3t^2\left(1+\frac{2}{3t}\right)}\times 3t^2$$
$$= 1$$
$$\therefore \frac{1}{3t^2+2t} \sim \frac{1}{3t^2}, (t\to\infty).$$

Example 5.1.4.

$$\lim_{t\to\infty}\frac{\sin ht}{e^t/2} = \lim_{t\to\infty}\left(\frac{e^t-e^{-t}}{2}\times\frac{2}{e^t}\right)$$
$$= \lim_{t\to\infty}\frac{\left(e^t-\frac{1}{e^t}\right)}{e^t}$$
$$= \lim_{t\to\infty}\left(1-\frac{1}{e^{2t}}\right)$$
$$= 1$$
$$\therefore \sin ht \sim \frac{e^t}{2}, (t\to\infty).$$

Definition 5.1.5. *If* $\lim_{t\to\infty}\frac{u(t)}{v(t)} = 0$*, then we say that "*$u(t)$ *is much smaller than* $v(t)$ *as* t *tends to infinity" and write*

$$u(t) << v(t), \quad (t\to\infty).$$

Note:1

$$y(t)\sim z(t), \quad (t\to\infty) \quad \textbf{iff} \quad y(t)-z(t) << z(t), \quad (t\to\infty).$$

For, suppose that $y(t)\sim z(t), (t\to\infty)$.

Then

$$\begin{aligned}\lim_{t\to\infty}\frac{y(t)-z(t)}{z(t)} &= \lim_{z\to\infty}\frac{y(t)}{z(t)} - \lim_{t\to\infty}\frac{z(t)}{z(t)}\\ &= 1-1\\ &= 0.\end{aligned}$$

$$\Rightarrow y(t)-z(t) << z(t), (t\to\infty).$$

Conversely, suppose $y(t)-z(t) << z(t), \quad (t\to\infty)$.

$$\begin{aligned}\Rightarrow \lim_{t\to\infty}\frac{y(t)-z(t)}{z(t)} &= 0\\ \Rightarrow \lim_{t\to\infty}\frac{y(t)}{z(t)}-1 &= 0\\ \Rightarrow \lim_{t\to\infty}\frac{y(t)}{z(t)} &= 1\end{aligned}$$
$$\Rightarrow y(t)\sim z(t), (t\to\infty).$$

Note: 2 If $y(t) \sim z(t), (t \to \infty)$ and if $u(t) << z(t)$, $(t \to \infty)$, then $y(t) + u(t) \sim z(t), (t \to \infty)$.

For,

$$\begin{aligned} \lim_{t\to\infty} \frac{y(t)+u(t)}{z(t)} &= \lim_{t\to\infty} \frac{y(t)}{z(t)} + \lim_{t\to\infty} \frac{u(t)}{z(t)} \\ &= 1+0 \\ &= 1. \end{aligned}$$

For example, we know that

$$\left(4t^2+t\right)^{3/2} \sim 8t^3 \quad , (t \to \infty).$$

Also,

$$\begin{aligned} \lim_{t\to\infty} \frac{t^2 \log t}{8t^3} &= \lim_{t\to\infty} \frac{\log t}{8t} \\ &= \lim_{t\to\infty} \frac{1/t}{8} \\ &= \lim_{t\to\infty} \frac{1}{8t} \\ &= 0 \end{aligned}$$

$$\therefore t^2 \log t << 8t^3, (t \to \infty).$$

Then

$$\left(4t^2+t\right)^{3/2} + t^2 \log t \sim 8t^3, (t \to \infty)$$

because

$$\begin{aligned} \lim_{t\to\infty} \frac{\left(4t^2+t\right)^{3/2} + t^2 \log t}{8t^3} &= \lim_{t\to\infty} \frac{\left(4t^2+t\right)^{3/2}}{8t^3} + \lim_{t\to\infty} \frac{t^2 \log t}{8t^3} \\ &= 1+0 \\ &= 1. \end{aligned}$$

Example 5.1.6. *Discuss the asymptotic behavior of* $\left(4t^2+t\right)^{3/2}$.

Solution: Since $4t^2$ is much larger than t, when t is large, a good approximation should be given by $\left(4t^2\right)^{3/2} = 8t^3$. The corresponding relative error is given by

$$\begin{aligned} \frac{\left(4t^2+t\right)^{3/2} - 8t^3}{8t^3} &= \frac{8t^3\left(1+\frac{1}{4t}\right)^{3/2} - 8t^3}{8t^3} \\ &= \left(1+\frac{1}{4t}\right)^{3/2} - 1. \end{aligned}$$

If $f(x) = (1+x)^{3/2}$, $[0, x]$, then by Mean value theorem, $\exists\, c \in (0, x)$ such that

$$\begin{aligned} \frac{f(x) - f(0)}{x - 0} &= f'(c) \\ \implies \frac{(1+x)^{3/2} - 1}{x} &= \frac{3}{2}(1+c)^{\frac{1}{2}} \\ \implies (1+x)^{3/2} &= 1 + \frac{3}{2}(1+c)^{1/2}x. \end{aligned}$$

Therefore,

$$\frac{\left(4t^2+t\right)^{3/2} - 8t^3}{8t^3} = \frac{3}{2}(1+c)^{1/2}\left(\frac{1}{4t}\right), \tag{5.1}$$

where $0 < c < \frac{1}{4t}$.

$$\therefore \lim_{t\to\infty} \frac{\left(4t^2+t\right)^{3/2} - 8t^3}{8t^3} = 0.$$

That is, the relative error goes to zero as $t \to \infty$. Also from (5.1),

$$\left| \frac{\left(4t^2+t\right)^{3/2} - 8t^3}{8t^3} \right| \le \frac{M}{t} \text{ for some } M > 0 \text{, and } t \ge 1.$$

Thus, we say that the relative error goes to zero as $t \to \infty$ at a rate proportional to $1/t$.

Definition 5.1.7. *If there are constants M and t_0 so that $|u(t)| \le M|v(t)|$ for $t \ge t_0$, then we say that "$u(t)$ is big oh of $v(t)$ as t tends to infinity" and write*

$$u(t) = \mathcal{O}(v(t)), \quad (t \to \infty).$$

Example 5.1.8. *By previous example, we have*

$$\begin{aligned} \frac{\left(4t^2+t\right)^{3/2} - 8t^3}{8t^3} &= \mathcal{O}\left(\frac{1}{t}\right), (t \to \infty) \\ \therefore \left(4t^2+t\right)^{3/2} &= 8t^3(1 + \mathcal{O}(1/t)), (t \to \infty). \end{aligned}$$

Let us find a better approximation than the above as follows:

$$\begin{aligned} \left(4t^2+t\right)^{3/2} &= 8t^3\left(1 + \frac{1}{4t}\right)^{3/2} \\ &= 8t^3\left[1 + \frac{3}{2}\left(\frac{1}{4t}\right) + \frac{3}{4}(1+d)^{-1/2}\frac{1}{2}\left(\frac{1}{4t}\right)^2\right] \\ &= 8t^3\left[1 + \frac{3}{8t} + \frac{3}{128}(1+d)^{-1/2}\frac{1}{t^2}\right] \\ &= 8t^3\left[1 + \frac{3}{8t} + \mathcal{O}\left(\frac{1}{t^2}\right)\right], \quad (t \to \infty). \end{aligned}$$

We have

$$\left(4t^2+t\right)^{3/2} = 8t^3+3t^2+\frac{3}{16}(1+d)^{-1/2}t$$
$$> 8t^3+3t^2$$
$$> 8t^3 \quad \forall t>0.$$

$\therefore$ *The* $\mathcal{O}\left(\frac{1}{t^2}\right)$ *estimate* $8t^3+3t^2$ *is closer to* $\left(4t^2+t\right)^{3/2}$ *than the* $\mathcal{O}\left(\frac{1}{t}\right)$ *estimate* $8t^3$ *for all* $t>0$.

(ie) $\mathcal{O}\left(\frac{1}{t^2}\right)$ *approximation is better than an* $\mathcal{O}\left(\frac{1}{t}\right)$ *approximation if* t *is sufficiently large.*

Example 5.1.9. *Consider the exponential integral.*

$$E_n(x)=\int_1^{\infty}\frac{e^{-xt}}{t^n}dt,(x>0).$$

We shall first investigate the asymptotic behavior of $E_n(x)$ as $x\to\infty$.

$$\text{Take } n=\frac{1}{t^n}, \qquad dv=e^{-xt}dt$$
$$\Longrightarrow \qquad dn=-nt^{-n-1}dt, \quad v=\frac{e^{-xt}}{-x}.$$
$$=\frac{-n}{t^{n+1}}dt$$

$$\therefore \int_1^{\infty}\frac{e^{-xt}}{t^n}dt=\left[\frac{-e^{-xt}}{xt^n}\right]_1^{\infty}+\int_1^{\infty}\frac{e^{-xt}}{x}\left(\frac{-n}{t^{n+1}}\right)dt \;=\frac{e^{-x}}{x}-\frac{n}{x}\int_1^{\infty}\frac{e^{-xt}}{t^{n+1}}dt$$

Now,

$$\int_1^{\infty}\frac{e^{-xt}}{t^{n+1}}dt\le\int_1^{\infty}e^{-xt}dt$$
$$\Longrightarrow \int_1^{\infty}\frac{e^{-xt}}{t^{n+1}}dt\leqslant\frac{e^{-x}}{x}.$$
$$\therefore \int_1^{\infty}\frac{e^{-xt}}{t^n}dt\leqslant\frac{e^{-x}}{x}-\frac{n}{x}\left(\frac{e^{-x}}{x}\right)$$
$$=\frac{e^{-x}}{x}\left(1+\left(\frac{-n}{x}\right)\right).$$

So,for each fixed n,

$$E_n(x)=\int_1^{\infty}\frac{e^{-xt}}{t^n}dt=\frac{e^{-x}}{x}\left(1+\mathcal{O}\left(\frac{1}{x}\right)\right),(x\to\infty).$$

By using integration by parts repeatedly, we get for each positive integer k,

$$E_n(x)=\frac{e^{-x}}{x}\left[1-\frac{n}{x}+\frac{n(n+1)}{x^2}-\cdots\cdots+(-1)^k\frac{n(n+1)\cdots(n+k-1)}{x^k}\right.$$
$$\left.+\mathcal{O}\left(\frac{1}{x^{k+1}}\right)\right],(x\to\infty).$$

The series in brackets is called an "asymptotic series" and it diverges for each x by the ratio test.

Let us investigate the asymptotic behavior of $E_n(x)$ for large n.

$$\begin{aligned}
\text{Take } u &= e^{-xt}, \quad dv = t^{-n}dt \\
\implies du &= -xe^{-xt}dt, \quad v = \frac{t^{-n+1}}{-n+1}. \\
\therefore E_n(x) &= \int_1^\infty \frac{e^{-xt}}{t^n}dt \\
&= \left[\frac{e^{-xt}\, t^{-n+1}}{-n+1}\right]_1^\infty - \int_1^\infty \frac{t^{-n+1}}{-n+1}\left(-x \cdot e^{-xt}\right) dt \\
&= \frac{e^{-x}}{n-1} - \int_1^\infty \frac{x}{n-1} \cdot \frac{e^{-xt}}{t^{n-1}}dt. \\
\text{Since } \int_1^\infty \frac{e^{-xt}}{t^{n-1}}dt &\leqslant \int_1^\infty \frac{e^{-x}}{t^{n-1}}dt \\
&= e^{-x}\int_1^\infty t^{1-n}dt \\
&= e^{-x}\left[\frac{t^{2-n}}{2-n}\right]_1^\infty \\
&= \frac{e^{-x}}{n-2}, \\
\text{we have } E_n(x) &\leq \frac{e^{-x}}{n-1} - \frac{x}{n-1}\left(\frac{e^{-x}}{n-2}\right) \\
&= \frac{e^{-x}}{n-1}\left[1 + \left(\frac{-x}{n-2}\right)\right] \\
&= \frac{e^{-x}}{n-1}\left[1 + \mathcal{O}\left(\frac{1}{n-2}\right)\right], \quad (n \to \infty),
\end{aligned}$$

where x is any fixed positive number. So, we can write

$$E_n(x) = \frac{e^{-x}}{n-1}\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right], \quad (n \to \infty).$$

A second integration by parts gives

$$\begin{aligned}
E_n(x) &= \frac{e^{-x}}{n-1}\left[1 - \frac{x}{n-2} + \mathcal{O}\left(\frac{1}{(n-2)(n-3)}\right)\right], \quad (n \to \infty) \\
&= \frac{e^{-x}}{n-1}\left[1 - \frac{x}{n-2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right], \quad (n \to \infty),
\end{aligned}$$

and the calculation can be continued to any number of terms.

Note: We could write $\mathcal{O}\left(\frac{1}{n}\right)$ instead of $\mathcal{O}\left(\frac{1}{n-2}\right)$, and $\mathcal{O}\left(\frac{1}{n^2}\right)$ instead of $\mathcal{O}\left(\frac{1}{(n-2)(n-3)}\right)$ without any loss of information.

Let Us Sum Up:

In this section, we have discussed the basic concepts and tools of asymptotic analysis. Moreover, some standard non-trivial examples were also given to understand the study on asymptotic behavior.

Check your Progress:

1. If $\lim_{t\to\infty} \frac{y(t)}{z(t)} = 1$, then we say that ....

   (A) $y(t)$ is asymptotic to $z(t)$ as $t \to \infty$

   (B) $y(t)$ is much smaller than $z(t)$ as $t \to \infty$

   (C) $y(t)$ is equivalent to $z(t)$ as $t \to \infty$

   (D) None of these

2. If $\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0$, then we can denote it by ....

   (A) $u(t) \sim v(t), \quad (t \to \infty)$ (B) $u(t) << v(t), \quad (t \to \infty)$

   (C $v(t) << u(t), \quad (t \to \infty)$ (D) None of these

3. If $y(t) \sim z(t), (t \to \infty)$ and if $u(t) << z(t)$, $(t \to \infty)$ then ....

   (A) $y(t) + u(t) << z(t), (t \to \infty)$ (B) $z(t) \sim y(t) + u(t), (t \to \infty)$

   (C $y(t) + u(t) \sim z(t), (t \to \infty)$ (D) None of these

5.2 Asymptotic Analysis of Sums

Example 5.2.1. *Find the asymptotic approximate solution of*

$$y_{n+1} - ny_n = 1, \quad n = 1, 2, 3 \ldots$$

Solution: W.K.T, the solution of the given equation is

$$y_n = (n-1)! \left[y_1 + \sum_{k=1}^{n-1} \frac{1}{k!} \right], n \geqslant 2.$$

Clearly, the sum in brackets is a partial sum of the Taylor series for e^θ with $\theta = 1$.

$$
\begin{aligned}
\therefore e &= \sum_{k=0}^{\infty} \frac{1}{k!} \\
&= \sum_{k=0}^{n-1} \frac{1}{k!} + \sum_{k=n}^{\infty} \frac{1}{k!} \\
&= 1 + \sum_{k=1}^{n-1} \frac{1}{k!} + \frac{e^c}{n!}, \quad (0 < c < 1). \quad \text{(By Taylor's formula)}
\end{aligned}
$$

$$
\Rightarrow \sum_{k=1}^{n-1} \frac{1}{k!} = e - 1 - \frac{e^c}{n!}.
$$

$$
\begin{aligned}
\text{Thus, } y_n &= (n-1)! \left[y_1 + e - 1 - \frac{e^c}{n!} \right] \\
&= (n-1)! \left[y_1 + e - 1 + 0\left(\frac{1}{n!}\right) \right], (n \to \infty).
\end{aligned}
$$

Thus, for large n, y_n is approximately $(y_1 + e - 1)(n-1)!$ and the relative error goes to zero like $\frac{1}{n!}$.

Example 5.2.2. *What is the asymptotic behavior of* $\sum_{k=1}^{n} k^k$ *?*

Solution:

$$
\begin{aligned}
\sum_{k=1}^{n} k^k &= 1 + 2^2 + 3^3 + 4^4 + \cdots + (n-1)^{n-1} + n^n \\
&= n^n \left[1 + \left\{ \frac{(n-1)^{n-1}}{n^n} + \frac{(n-2)^{n-2}}{n^n} + \cdots + \frac{2^2}{n^n} + \frac{1}{n^n} \right\} \right] \\
&= n^n \left[1 + \left\{ \left(\frac{n-1}{n}\right)^{n-1} \frac{1}{n} + \left(\frac{n-2}{n}\right)^{n-2} \frac{1}{n^2} + \cdots + \left(\frac{1}{n}\right) \frac{1}{n^{n-1}} \right\} \right] \\
&< n^n \left[1 + \left\{ \frac{1}{n} + \frac{1}{n^2} + \cdots + \frac{1}{n^{n-1}} \right\} \right].
\end{aligned}
$$

Now,

$$\begin{aligned}
\frac{1}{n}+\frac{1}{n^2}+\cdots+\frac{1}{n^{n-1}} &= \frac{1}{n}\left[1+\frac{1}{n}+\frac{1}{n^2}+\cdots+\frac{1}{n^{n-2}}\right] \\
&= \frac{1}{n}\sum_{k=0}^{n-2}\frac{1}{n^k} \\
&= \frac{1}{n}\sum_{k=0}^{n-2}\left(\frac{1}{n}\right)^k \\
&= \frac{1}{n}\left(\frac{1-(1/n)^{n-1}}{1-1/n}\right).
\end{aligned}$$

$$\implies \sum_{k=1}^{n} k^k < n^n\left[1+\frac{1}{n}\left(\frac{1-\left(\frac{1}{n}\right)^{n-1}}{1-1/n}\right)\right].$$

$$\therefore \sum_{k=1}^{n} k^k = n^n[1+\mathcal{O}(1/n)], (n\to\infty)$$

$$\left(\because \frac{1-(1/n)^{n-1}}{1-1/n} \text{ is bounded }\right)$$

Hence, the asymptotic value of $\sum_{k=1}^{n} k^k$ is given by the largest term n^n with a relative error that approaches zero like $\frac{1}{n}$ as $n\to\infty$.

Similarly, we can prove that

$$\sum_{k=1}^{n} k^k = n^n\left[1+\left(\frac{n-1}{n}\right)^{n-1}\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right], (n\to\infty).$$

Here the two largest terms of the series gives an $\mathcal{O}\left(\frac{1}{n^2}\right)$ asymptotic estimate.

Example 5.2.3. *Discuss the asymptotic behavior of*

$$\sum_{k=2}^{n-1} 2^k \log k.$$

Solution: W.K.T, the Abel's summation formula is given by

$$\sum_{k=m}^{n-1} a_k b_k = b_n\sum_{k=m}^{n-1} a_k - \sum_{k=m}^{n-1}\left(\sum_{i=m}^{k} a_i\right)\Delta b_k.$$

Putting $a_k = 2^k$ and $b_k = \log k$ in the above formula, we have

$$\begin{aligned}
\sum_{k=2}^{n-1} 2^k \log k &= \log n\sum_{k=2}^{n-1} 2^k - \sum_{k=2}^{n-1}\left(\sum_{i=2}^{k} 2^i\right)\Delta\log k \\
&= \log n\left(2^2+2^3+\cdots+2^{n-1}\right) - \sum_{k=2}^{n-1}\left(2^2+2^3+\cdots+2^k\right)\Delta\log k
\end{aligned}$$

$$= (\log n)\left(2^n - 4\right) - \sum_{k=2}^{n-1}\left(2^{k+1} - 4\right)\Delta \log k$$

$$= 2^n \log n - \sum_{k=2}^{n-1} 2^{k+1}\Delta \log k - 4\log n + 4\sum_{k=2}^{n-1}\Delta \log k.$$

Since the first two terms are asymptotically much larger than the last two terms, we will neglect them.

By the mean value theorem,

$$\begin{aligned}
\Delta \log k &< \frac{1}{k}. \\
\therefore \sum_{k=2}^{n-1} 2^{k+1}\Delta \log k &< \sum_{k=2}^{n-1} 2^{k+1}\left(\frac{1}{k}\right) \\
&= 2^3\left(\frac{1}{2}\right) + 2^4\left(\frac{1}{3}\right) + 2^5\left(\frac{1}{4}\right) + \cdots + 2^{n-1} \\
&\quad \left(\frac{1}{n-2}\right) + 2^n\left(\frac{1}{n-1}\right) \\
&= \frac{2^n}{n-1}\left[1 + \frac{n-1}{n-2}\cdot\frac{1}{2} + \frac{n-1}{n-3}\cdot\frac{1}{2^2} + \cdots + \frac{n-1}{2}\cdot\frac{1}{2^{n-3}}\right] \\
&= \frac{2^n}{n-1}\sum_{k=0}^{n-3}\frac{n-1}{n-(k+1)}\cdot\frac{1}{2^k}.
\end{aligned}$$

We can easily verify that $\frac{n-1}{n-(k+1)} \leqslant k+1$, when $0 \leqslant k \leqslant n-3$.

$$\therefore \sum_{k=2}^{n-1} 2^{k+1}\Delta \log k < \frac{2^n}{n-1}\sum_{k=0}^{n-3}\frac{k+1}{2^k}.$$

By the ratio test, $\sum_{k=0}^{n-3}\frac{k+1}{2^k}$ is bounded.

$$\begin{aligned}
\therefore \sum_{k=2}^{n-1} 2^{k+1}\Delta \log k &< M\left(\frac{2^n}{n-1}\right), M > 0 \text{ is a constant.} \\
\text{Thus, } \sum_{k=2}^{n-1} 2^k \log k &= 2^n \log n + 2^n \mathcal{O}\left(\frac{1}{n-1}\right) \\
&= 2^n \log n\left[1 + \mathcal{O}\left(\frac{1}{(n-1)\log n}\right)\right], (n \to \infty).
\end{aligned}$$

Here, the asymptotic behavior of the sum is given not by the largest term but rather by twice the largest term.

Example 5.2.4. *Discuss the asymptotic behavior of* $\sum_{k=1}^{n} \log k$.

Solution: W.K.T, the Euler summation formula for $i \geq 1$ and $n \geq 2$ is given by

$$\begin{aligned}
\sum_{k=1}^{n} y(k) &= \int_1^n y(x)dx + \frac{y(n)+y(1)}{2} + \sum_{j=1}^{i} \frac{B_{2j}}{(2j)!}\left[y^{(2j-1)}(n) - y^{(2j-1)}(1)\right] \\
&\quad -\frac{1}{(2i)!}\int_1^n y^{(2i)}(x)B_{2i}(x-[x])dx. \\
\therefore \sum_{k=1}^{n} \log k &= \int_1^n \log x dx + \frac{\log n + \log 1}{2} + \sum_{j=1}^{i} \frac{B_{2j}}{(2j)!}\Big[\log^{(2j-1)}(n) - \log^{(2j-1)}(1)\Big] \\
&\quad -\frac{1}{(2i)!}\int_1^n \log^{(2i)} x B_{2i}(x-[x])dx \\
&= \Big[x\log x - x\Big]_1^n + \frac{\log n}{2} + \sum_{j=1}^{i} \frac{B_{2j}}{(2j)!}\left[\frac{(2j-2)!}{n^{2j-1}} - \frac{(2j-2)!}{1}\right] \\
&\quad +\frac{1}{(2i)!}\int_1^n \frac{(2i-1)!}{x^{2i}} B_{2i}(x-[x])dx \\
&= n\log n - n + 1 + \frac{\log n}{2} + \sum_{j=1}^{i} \frac{B_{2j}}{(2j)\cdot(2j-1)}\cdot\frac{1}{n^{2j-1}} - \sum_{j=1}^{i}\frac{B_{2j}}{(2j)(2j-1)} \\
&\quad +\frac{1}{2i}\int_1^n \frac{B_{2i}(x-[x])}{x^{2i}}dx \\
\text{That is,} \sum_{k=1}^{n} \log k &= n\log n - n + \frac{\log n}{2} + \sum_{j=1}^{i} \frac{B_{2j}}{(2j)(2j-1)}\cdot\frac{1}{n^{2j-1}}
\end{aligned}$$

$$+\left\{1 - \sum_{j=1}^{i}\frac{B_{2j}}{(2j)(2j-1)} + \frac{1}{2i}\int_1^\infty \frac{B_{2i}(x-[x])}{x^{2i}}dx\right\} - \frac{1}{2i}\int_n^\infty \frac{B_{2i}(x-[x])}{x^{2i}}dx. \tag{5.2}$$

The expression in braces is independent of n, so we take it as $\gamma(i)$.

$\therefore$ By eq. (5.2),

$$\gamma(i) = \sum_{k=1}^{n} \log k - n\log n + n - \frac{\log n}{2} - \sum_{j=1}^{i} \frac{B_{2j}}{(2j)(2j-1)} \cdot \frac{1}{n^{2j-1}} + \frac{1}{2i}\int_{n}^{\infty} \frac{B_{2i}(x-[x])}{x^{2i}}dx.$$

$$\text{Then } \gamma(i+1) = \sum_{k=1}^{n} \log k - n\log n + n - \frac{\log n}{2} - \sum_{j=1}^{i+1} \frac{B_{2j}}{(2j)(2j-1)} \cdot \frac{1}{n^{2j-1}} + \frac{1}{2i+2}\int_{n}^{\infty} \frac{B_{2i+2}(x-[x])}{x^{2i+2}}dx.$$

$$\therefore \gamma(i+1) - \gamma(i) = -\frac{B_{2i+2}}{(2i+2)(2i+1)} \cdot \frac{1}{n^{2i+1}} + \int_{n}^{\infty} \left(\frac{B_{2i+2}(x-[x])}{(2i+2)x^{2i+2}} - \frac{B_{2i}(x-[x])}{(2i)x^{2i}} \right) dx \to 0 \text{ as } n \to \infty.$$

So, γ is independent of i.

Equation eq. (5.2) gives asymptotic estimates of $\sum_{k=1}^{n} \log k$ for each $i = 1, 2, \ldots$.

For $i = 2$, we have

$$\sum_{k=1}^{n} \log k = n\log n - n + \frac{\log n}{2} + \frac{B_2}{2n} + \frac{B_4}{12n^3} + \gamma - \frac{1}{4}\int_{n}^{\infty} \frac{B_4(x-[x])}{x^4}dx$$
$$= n\log n - n + \frac{\log n}{2} + \frac{1}{12n} + \gamma + \mathcal{O}\left(\frac{1}{n^3}\right), \quad (n \to \infty) \quad (\because B_2 = \frac{1}{6}). \tag{5.3}$$

Now, applying exponential function on both sides, we get

$$n! = e^{\gamma} \left(\frac{n}{e}\right)^n \sqrt{n} \quad e^{\frac{1}{12n} + \mathcal{O}\left(\frac{1}{n3}\right)}, (n \to \infty)$$

or by Taylor's formula,

$$n! = e^{\gamma} \left(\frac{n}{e}\right)^n \sqrt{n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \mathcal{O}\left(\frac{1}{n3}\right)\right), \quad (n \to \infty). \tag{5.4}$$

To find γ, consider the Walli's formula

$$\frac{\pi}{2} = \lim_{n\to\infty} \left[\frac{2.4 \cdot \cdots \cdot 2n}{1 \cdot 3 \cdots (2n-1)}\right]^2 \frac{1}{2n+1}.$$

Now,

$$\begin{aligned}
\left[\frac{2.4\cdot\dots\cdot 2n}{1.3\cdots(2n-1)}\right]^2\frac{1}{2n+1} &= \left(\frac{2.4\cdots 2n}{1.3\cdots(2n-1)}\cdot\frac{2.4\cdots 2n}{2.4\cdots 2n}\right)^2\frac{1}{2n+1}\\
&= \left[\frac{(2^n)\,(2^n)\,n!n!}{(2n)!}\right]^2\frac{1}{2n+1}\\
&= \frac{2^{4n}(n!)^4}{((2n)!)^2}.\frac{1}{2n+1}\\
&= \frac{2^{4n}}{2n+1}.\frac{e^{4\gamma}\left(\frac{n}{e}\right)^{4n}n^2\left(1+\frac{1}{12n}+\frac{1}{288n^2}+0\left(\frac{1}{n^3}\right)\right)^4}{e^{2\gamma}\left(\frac{2n}{e}\right)^{4n}2n\left(1+\frac{1}{24n}+\frac{1}{4\times 288n^2}+0\left(\frac{1}{n^3}\right)\right)^2}\\
&= \frac{ne^{2\gamma}}{2(2n+1)}\frac{\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right)^4}{\left(1+\frac{1}{24n}+\frac{1}{4\times 288n^2}+\mathcal{O}\left(\frac{1}{n3}\right)\right)^2},(n\to\infty)\\
&= \frac{e^{2\gamma}}{4\left(1+\frac{1}{2n}\right)}\frac{\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right)^4}{\left(1+\frac{1}{24n}+\frac{1}{4\times 288n^2}+\mathcal{O}\left(\frac{1}{n3}\right)\right)^2},(n\to\infty).
\end{aligned}$$

$$\begin{aligned}
\implies \lim_{n\to\infty}\left(\frac{\left(\frac{2.4\ldots 2n}{1.3\ldots.(2n-1)}\right)^2\frac{1}{2n+1}}{\frac{e^{2\gamma}}{4(1+\frac{1}{2n})}\frac{\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right)^4}{\left(1+\frac{1}{24n}+\frac{1}{4\times 288n^2}+\mathcal{O}\left(\frac{1}{n3}\right)\right)^2}}\right) &= 1.\\
\implies \lim_{n\to\infty}\left(\frac{\left(\frac{2.4\ldots 2n}{1.3\ldots.(2n-1)}\right)^2\frac{1}{2n+1}}{\frac{e^{2\gamma}}{4}}\right) &= 1.\\
\Rightarrow \left(\frac{2\cdot 4\cdots 2n}{1\cdot 3\ldots(2n-1)}\right)^2\frac{1}{2n+1} &\sim \frac{e^{2\gamma}}{4},(n\to\infty).
\end{aligned}$$

Hence, by Wallis formula,

$$\begin{aligned}
\frac{e^{2\gamma}}{4} &= \frac{\pi}{2}\\
\implies e^{2\gamma} &= 2\pi\\
\implies e^{\gamma} &= \sqrt{2\pi}.
\end{aligned}$$

$\therefore$ Eq. (5.4) becomes

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\frac{1}{12n}+\frac{1}{288n^2}+\mathcal{O}\left(\frac{1}{n^3}\right)\right),(n\to\infty). \tag{5.5}$$

This equation is called Stirling's formula and it gives a good estimate for $n!$.

Note: The Stirling's formula is valid for gamma function also.

$$(i.e,)\ \Gamma(t+1) = \sqrt{2\pi t}\left(\frac{t}{e}\right)^t\left(1+\frac{1}{12t}+\frac{1}{288t^2}+\mathcal{O}\left(\frac{1}{t^3}\right)\right),(t\to\infty) \tag{5.6}$$

Example 5.2.5. *Discuss the asymptotic behavior of solution of the first-order linear equation* $t\Delta u(t) - \frac{1}{2}u(t) = 0$.

Solution: Consider

$$t\Delta u(t) - \frac{1}{2}u(t) = 0.$$

$$\implies t(u(t+1) - u(t)) - \frac{1}{2}u(t) = 0$$
$$\implies t(u(t+1)) = \left(t + \frac{1}{2}\right)u(t)$$
$$\Rightarrow u(t+1) = \frac{(t+\frac{1}{2})}{t}u(t).$$

Its general solution is given by

$$u(t) = \frac{c\Gamma(t+\frac{1}{2})}{\Gamma(t)}, \quad c \text{ is a constant.}$$
$$\Rightarrow u(t) = \frac{c\Gamma\big((t+\frac{1}{2})+1\big)}{t+\frac{1}{2}} \cdot \frac{t}{\Gamma(t+1)}$$
$$= \frac{c}{\left(1+\frac{1}{2t}\right)} \frac{\Gamma\big((t+\frac{1}{2})+1\big)}{\Gamma(t+1)}$$

$$\Rightarrow u(t) = \frac{c}{\left(1+\frac{1}{2t}\right)} \frac{\sqrt{2\pi(t+\frac{1}{2})}\left(\frac{t+\frac{1}{2}}{e}\right)^{t+\frac{1}{2}}}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^t} \frac{\left(1+\frac{1}{12(t+\frac{1}{2})}+\frac{1}{288(t+\frac{1}{2})^2}+\mathcal{O}\left(\frac{1}{t^3}\right)\right)}{\left(1+\frac{1}{12t}+\frac{1}{288t^2}+\mathcal{O}\left(\frac{1}{t^3}\right)\right)}, (t\to\infty).$$

$$\Rightarrow u(t) \sim c\frac{\sqrt{2\pi\left(t+\frac{1}{2}\right)}\left(\frac{t+\frac{1}{2}}{e}\right)^{t+\frac{1}{2}}}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^t}, \quad (t\to\infty).$$

Now,

$$\frac{\sqrt{2\pi\left(t+\frac{1}{2}\right)}\left(\frac{t+\frac{1}{2}}{e}\right)^{t+\frac{1}{2}}}{\sqrt{2\pi t}\left(\frac{t}{e}\right)^t} = \frac{(t+\frac{1}{2})^{\frac{1}{2}}(t+\frac{1}{2})^{t+\frac{1}{2}}}{t^{\frac{1}{2}}\left(\frac{t}{e}\right)^t e^{t+\frac{1}{2}}}$$
$$= \frac{(t+\frac{1}{2})^{\frac{1}{2}+t+\frac{1}{2}}}{t^{t+\frac{1}{2}}\sqrt{e}}$$

$$= \left(\frac{t+\frac{1}{2}}{t}\right)^t\left(\frac{t+\frac{1}{2}}{t^{\frac{1}{2}}}\right)\frac{1}{\sqrt{e}}$$
$$= \left(\frac{t+\frac{1}{2}}{t}\right)^t\left(\frac{t(1+\frac{1}{2t})}{t^{\frac{1}{2}}}\right)\cdot\frac{1}{\sqrt{e}}$$
$$= \left(\frac{t+\frac{1}{2}}{t}\right)^t.\left(1+\frac{1}{2t}\right)\frac{t^{\frac{1}{2}}}{\sqrt{e}}$$

Thus, using the fact that $\lim_{t\to\infty}\left(\frac{t+\frac{1}{2}}{t}\right)^t = \sqrt{e}$, we get

$$u(t) \sim c \cdot t^{\frac{1}{2}}, (t \to \infty).$$

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Asymptotic approximations of sums for large n
2. Stirling's formula
3. Use of Euler summation formula in establishing asymptotic behavior

Check your Progress:

1. We can write $\mathcal{O}\left(\frac{1}{n}\right)$ instead of ....

   (A) $\mathcal{O}\left(\frac{1}{n^2}\right)$ (B) $\mathcal{O}\left(\frac{1}{n(n-1)}\right)$ (C) $\mathcal{O}\left(\frac{1}{n-3}\right)$ (D) None of these

2. If there are constants M and t_0 so that $|u(t)| \leq M|v(t)|$ for $t \geq t_0$, then we can denote it by ....

   (A) $u(t) \sim v(t), \quad (t \to \infty)$ (B) $u(t) << v(t), \quad (t \to \infty)$

   (C) $v(t) << u(t), \quad (t \to \infty)$ (D) $u(t) = \mathcal{O}(v(t)), \quad (t \to \infty)$

3. $\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{2.4\cdots 2n}{1\cdot 3\cdots(2n-1)}\right]^2 \frac{1}{2n+1}$ is known as ....

   (A) Taylor's formula (B) Abel's formula

   (C) Walli's formula (D) Stirling's formula

5.3 Linear Equations

This section introduces the study of the asymptotic behavior of solutions of homogeneous linear equations.

Let $u(t)$ be any nontrivial solution of the equation

$$u(t+2) + p_1 u(t+1) + p_0 u(t) = 0,$$

where p_0, p_1 are constants and the characteristic roots λ_1, λ_2 satisfy $|\lambda_1| > |\lambda_2|$. Then $u(t) = a\lambda_1^t + b\lambda_2^t$ for some constants a, b.

If $a \neq 0$, then

$$\begin{aligned} \frac{u(t+1)}{u(t)} &= \frac{a\lambda_1^{t+1} + b\lambda_2^{t+1}}{a\lambda_1^t + b\lambda_2^t} \\ &= \frac{\lambda_1\left(1 + \frac{b}{a}\lambda_1\left(\frac{\lambda_2}{\lambda_1}\right)^{t+1}\right)}{1 + \frac{b}{a}\left(\frac{\lambda_2}{\lambda_1}\right)^t} \to \lambda_1, \quad (t \to \infty). \end{aligned}$$

If $a = 0$, then

$$\frac{u(t+1)}{u(t)} = \frac{b\lambda_2^{t+1}}{b\lambda_2^t} = \lambda_2.$$

So, in any case the ratio $\frac{u(t+1)}{u(t)}$ converges to a root of the characteristic equation as t goes to infinity.

If $|\lambda_1| = |\lambda_2|$, this property may fail. The equation

$$u(t+2) - u(t) = 0$$

has characteristic roots $\lambda = \pm 1$ (so $|\lambda_1| = |\lambda_2|$), and for the solution $u(t) = 2 + (-1)^t$, we find

$$\frac{u(t+1)}{u(t)} = \frac{2 + (-1)^{t+1}}{2 + (-1)^t}.$$

This expression produces a sequence that alternates between 3 and $\frac{1}{3}$.

Definition 5.3.1. *A homogeneous linear equation*

$$u(t+n) + p_{n-1}(t)u(t+n-1) + \cdots + p_0(t)u(t) = 0 \tag{5.7}$$

is said to be of "Poincaré type" if $\lim_{t\to\infty} p_k(t) = p_k$ *for* $k = 0, 1, \cdots, n-1$ *(i.e., if the coefficient functions converge to constants as* t *goes to infinity).*

Theorem 5.3.2. ***Poincaré's Theorem***

If Eq. (5.7) is of Poincaré type and if the roots $\lambda_1, \cdots, \lambda_n$ *of* $\lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_0 = 0$ *satisfy* $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$*, then every nontrivial solution* u *of Eq. (5.7) satisifes*

$$\lim_{t\to\infty} \frac{u(t+1)}{u(t)} = \lambda_i$$

for some i*.*

Proof. Since the main ideas of the proof are evident in the case $n = 2$, we consider that case only and write Eq. (5.7) in the form

$$u(t+2) + (a + \alpha(t))u(t+1) + (b + \beta(t))u(t) = 0, \tag{5.8}$$

where $\alpha(t), \beta(t) \to 0$ as $t \to \infty$. Recall that the roots λ_1, λ_2 of the characteristic equation $\lambda^2 + a\lambda + b = 0$ satisfy $|\lambda_1| > |\lambda_2|$.

Let $u(t)$ be a nontrivial solution of (5.8) and let $x(t), y(t)$ be chosen to satisfy

$$\begin{aligned} x(t) + y(t) &= u(t) \\ \lambda_1 x(t) + \lambda_2 y(t) &= u(t+1). \end{aligned} \tag{5.9}$$

The system (5.9) has for each t a unique nontrivial solution since

$$\det \begin{bmatrix} 1 & 1 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \lambda_2 - \lambda_1 \neq 0$$

and either $u(t)$ or $u(t+1)$ is not zero.

Using Eqs. (5.8) and (5.9), we arrive at the system

$$x(t+1) = \lambda_1 x(t) + (\lambda_2 - \lambda_1)^{-1} \left[\left[(\lambda_1 \alpha(t) + \beta(t))\, x(t) \right] + \left[(\lambda_2 \alpha(t) + \beta(t))\, y(t) \right] \right], \tag{5.10}$$

$$y(t+1) = \lambda_2 y(t) + (\lambda_1 - \lambda_2)^{-1} \left[\left[(\lambda_2 \alpha(t) + \beta(t))\, y(t) \right] + \left[(\lambda_1 \alpha(t) + \beta(t))\, x(t) \right] \right]. \tag{5.11}$$

Choose $\epsilon > 0$ small enough that $\frac{|\lambda_2| + \epsilon}{|\lambda_1| - \epsilon} < 1$, and choose N so large that

$$|\lambda_1 - \lambda_2|^{-1} |\lambda_i \alpha(t) + \beta(t)| < \frac{\epsilon}{2}, \quad (i = 1, 2)$$

if $t \geq N$.

Let $t \geq N$ and suppose $|x(t)| \geq |y(t)|$.

From Eq. (5.10),

$$\begin{aligned} |x(t+1)| &\geq |\lambda_1| \, |x(t)| - \frac{\epsilon}{2}(|x(t)| + |y(t)|) \\ &\geq (|\lambda_1| - \epsilon) \, |x(t)|. \end{aligned}$$

From Eq. (5.11),

$$\begin{aligned} |y(t+1)| &\leq |\lambda_2| \, |y(t)| + \frac{\epsilon}{2}(|y(t)| + |x(t)|) \\ &\leq (|\lambda_2| + \epsilon) \, |x(t)|. \end{aligned}$$

Taking a ratio of these inequalities, we have

$$\left|\frac{y(t+1)}{x(t+1)}\right| \leq \frac{|\lambda_2| + \epsilon}{|\lambda_1| - \epsilon} < 1.$$

So, $|x(t+1)| > |y(t+1)|$, and inductively we conclude that $|x(s)| > |y(s)|$ for all $s > t$. Consequently, there is a number $M \geq N$ so that either

$$|x(t)| > |y(t)| \quad \text{for } t \geq M \tag{5.12}$$

or

$$|y(t)| > |x(t)| \quad \text{for } t \geq M. \tag{5.13}$$

Suppose that Eq. (5.12) is true. There is a number r in $[0, 1]$ (the "limit superior") so that for each $\delta > 0$

$$\left|\frac{y(t)}{x(t)}\right| < r + \delta \tag{5.14}$$

for sufficiently large t, and

$$\left|\frac{y(t)}{x(t)}\right| > r - \delta \tag{5.15}$$

for infinitely many values of t.

From Eqs. (5.11) and (5.10),

$$|y(t+1)| \leq |\lambda_2|\,|y(t)| + \epsilon|x(t)|$$

$$|x(t+1)| \geq |\lambda_1|\,|x(t)| - \epsilon|x(t)|$$

for $t \geq M$, so by Eq. (5.15)

$$r - \delta < \left|\frac{y(t+1)}{x(t+1)}\right| \leq \frac{|\lambda_2|\left|\frac{y(t)}{x(t)}\right| + \epsilon}{|\lambda_1| - \epsilon}$$

for infinitely many values of t. By Eq. (5.14)

$$r - \delta < \frac{|\lambda_2|\,(r + \delta) + \epsilon}{|\lambda_1| - \epsilon}$$

or

$$r < \frac{\delta\,(|\lambda_1| + |\lambda_2| - \epsilon) + \epsilon}{|\lambda_1| - |\lambda_2| - \epsilon}.$$

Since ϵ and δ may be chosen as small as we like, it follows that $r = 0$. Thus if Eq. (5.12) is true, then $\lim_{t\to\infty} \frac{y(t)}{x(t)} = 0$.

From Eq. (5.9),

$\frac{u(t)}{x(t)} \to 1$ and $\frac{u(t+1)}{x(t)} \to \lambda_1$ as $t \to \infty$. Then $\frac{u(t+1)}{u(t)} \to \lambda_1$ as $t \to \infty$.

In a similar way, Eq. (5.13) implies

$\lim_{t\to\infty} \frac{x(t)}{y(t)} = 0$ and $\lim_{t\to\infty}(u(t+1)/u(t)) = \lambda_2$. □

Theorem 5.3.3. ***Perron's Theorem***

In addition to the assumptions of Theorem 5.3.2 , suppose that $p_0(t) \neq 0$ for each t. Then there are n independent solutions $u_1, \cdots, u_n$ of Eq. (5.7) that satisfy

$$\lim_{t\to\infty} \frac{u_i(t+1)}{u_i(t)} = \lambda_i, \quad (i = 1, \cdots, n).$$

Example 5.3.4. *Consider $(t+2)u(t+2) - (t+3)u(t+1) + 2u(t) = 0$.*

Dividing throughout by $t+2$, we obtain

$$u(t+2) - \frac{t+3}{t+2}u(t+1) + \frac{2}{t+2}u(t) = 0,$$

and the equation is of Poincare type since $\frac{t+3}{t+2} \to 1$ and $\frac{2}{t+2} \to 0$ as $t \to \infty$.

The associated characteristic equation is $\lambda^2 - \lambda = 0$, so $\lambda_1 = 1$ and $\lambda_2 = 0$.

By Perron's Theorem, there are independent solutions u_1, u_2 so that

$\frac{u_1(t+1)}{u_1(t)} \to 1$, $\frac{u_2(t+1)}{u_2(t)} \to 0$ *as* $t \to \infty$.

Remark 5.3.5. *For most purposes, we would like to have information about the asymptotic behavior of the solutions themselves. Knowing the limiting value of $\frac{u(t+1)}{u(t)}$ gives partial information but does not immediately yield an asymptotic approximation for $u(t)$. For example, some of the functions that satisfy $\lim_{t\to\infty} \frac{u(t+1)}{u(t)} = 1$ are $u(t) = 5, t$, $3t^2 + 12, t^{67}, e^{\sqrt{t}}, e^{-\sqrt{t}}, \frac{1}{t^3-7}, \log t$, etc.*

Theorem 5.3.6. *Suppose $\frac{u(t+1)}{u(t)} \to \lambda \quad (t \to \infty)$.*

(a) If $\lambda \neq 0$, then $u(t) = \pm\lambda^t e^{z(t)}$ with $z(t) << t, \quad (t \to \infty)$.

(b) If $\lambda = 0$, then $|u(t)| = e^{-z(t)}$ with $z(t) >> t, \quad (t \to \infty)$.

Proof. Let $v(t) = \left|\frac{u(t)}{\lambda^t}\right|$. Then

$$\frac{v(t+1)}{v(t)} = \left|\frac{\frac{u(t+1)}{\lambda^t+1}}{\frac{u(t)}{\lambda^t}}\right| = \left|\frac{1}{\lambda}\frac{u(t+1)}{u(t)}\right| \to 1, \quad (t \to \infty).$$

Since $v(t)$ is positive for t sufficiently large, we can let $z(t) = \log v(t)$. Then

$$z(t+1) - z(t) = \log \frac{v(t+1)}{v(t)} \to 0, \quad (t \to \infty).$$

Let $\epsilon > 0$ and choose m so that $|z(t+1) - z(t)| < \epsilon$ for all $t > m$. For $t > m$,

$$\begin{aligned} |z(t) - z(m)| &\le \sum_{k=m+1}^{t} |z(k) - z(k-1)| \\ &< \epsilon(t - m. \end{aligned}$$

So

$$|z(t)| < \epsilon(t-m) + |z(m)|$$

or

$$\begin{aligned} \left|\frac{z(t)}{t}\right| &< \epsilon\left(1 - \frac{m}{t}\right) + \left|\frac{z(m)}{t}\right| \\ &< 2\epsilon \end{aligned}$$

for t sufficiently large. Since $\epsilon > 0$ was arbitrary, $z(t) << t, (t \to \infty)$, and the proof of (a) is complete.

The proof of part (b) is similar. □

Remark 5.3.7. *If $\lambda = 0$, then Theorem 5.3.6(b) implies that $u(t)$ must tend to zero faster than e^{-ct} for every positive constant c. For $\lambda > 0$, Theorem 5.3.6(a) is equivalent to the statement that $(\lambda - \delta)^t \ll |u(t)| \ll (\lambda + \delta)^t, (t \to \infty)$ for each small $\delta > 0$.*

Example 5.3.8. *We know that one solution of*

$$(t+2)u(t+2) - (t+3)u(t+1) + 2u(t) = 0.$$

is $u(t) = \frac{2^t}{t!}$. Note that $\frac{u(t+1)}{u(t)} = \frac{2}{t+1} \to 0$ as $t \to \infty$, so we can take $u_2(t) = \frac{2^t}{t!}$. Let's try to produce more information about $u_1(t)$. We know that

$$\frac{u_1(t+1)}{u_1(t)} = 1 + \varphi(t), \tag{5.16}$$

where $\varphi(t) \to 0$ as $t \to \infty$. Writing the difference equation in the form

$$(t+2)\frac{u_1(t+2)}{u_1(t+1)} - (t+3) + 2\frac{u_1(t)}{u_1(t+1)} = 0$$

and substituting Eq. (5.16), we have

$$(t+2)(1 + \varphi(t+1)) - (t+3) + \frac{2}{1+\varphi(t)} = 0.$$

By the Mean Value Theorem (applied to the function $\frac{2}{1+u}$),

$$\frac{2}{1+\varphi(t)} = 2 + \mathcal{O}(\varphi(t)), \quad (t \to \infty).$$

So we have

$$(t+2)(1+\varphi(t+1)) - (t+3) + 2 + \mathcal{O}(\varphi(t)) = 0.$$

Rearranging,

$$\varphi(t+1) = -\frac{1}{t+2} + \mathcal{O}\left(\frac{\varphi(t)}{t}\right), \quad (t \to \infty).$$

We conclude that

$$\varphi(t) = -\frac{1}{t+1} + \mathcal{O}\left(\frac{1}{t^2}\right), \quad (t \to \infty).$$

Substitute this last expression into Eq.(5.16) to obtain

$$u_1(t+1) = \frac{t}{t+1}\left(1 + \mathcal{O}\left(\frac{1}{t^2}\right)\right) u_1(t).$$

We solve this equation by iteration, beginning with a value $t = t_0$ so that $u_1(t_0) \neq 0$ and $1 + \mathcal{O}\left(\frac{1}{t^2}\right) > 0$ for $t \geq t_0$:

$$\begin{aligned} u_1(t) &= \prod_{s=t_0}^{t-1} \frac{s}{s+1} \prod_{s=t_0}^{t-1}\left(1 + \mathcal{O}\left(\frac{1}{s^2}\right)\right) u_1(t_0) \\ &= \frac{t_0}{t} u_1(t_0) \prod_{s=t_0}^{t-1}\left(1 + \mathcal{O}\left(\frac{1}{s^2}\right)\right). \end{aligned}$$

In order to complete the calculation, we need the following theorem.

Theorem 5.3.9. *Assume that both $\sum_{s=t_0}^{\infty} a_s$ and $\sum_{s=t_0}^{\infty} a_s^2$ converge and $1 + a_s > 0$ for $s \geq t_0$. Then $\lim_{t\to\infty} \prod_{s=t}^{t-1}(1+a_s)$ exists and is equal to a positive constant.*

Returning to our calculation, we see that $\lim_{t\to\infty} t u_1(t) = C \neq 0$, so we finally have

$$u_1(t) \sim \frac{C}{t}, \quad (t \to \infty).$$

Note: An equation that is not of Poincaré type can be converted to one of Poincaré type by a change of variable.

Example 5.3.10. *Consider* $\quad u(t+2) - (t+1)u(t+1) + u(t) = 0.$

If this equation has a solution that increases rapidly as t increases, then the terms $u(t+2)$ and $(t+1)u(t+1)$ will increase more rapidly than the term $u(t)$, so

$$u(t+2) \sim (t+1)u(t+1), \quad (t \to \infty).$$

This relation suggests that $u(t)$ may grow as $(t-1)$ does! Consequently, we factor off this behavior by making the change of variable

$$u(t) = (t-1)!v(t).$$

The resulting equation for v is

$$v(t+2) - v(t+1) + \frac{v(t)}{t(t+1)} = 0$$

which is of Poincaré type with characteristic roots $\lambda = 0, 1$. As in the previous example, set

$$\frac{v(t+1)}{v(t)} = 1 + \varphi(t),$$

where $\varphi(t) \to 0$ as $t \to \infty$, and substitution yields an equation for φ :

$$\varphi(t+1) + \frac{1}{t(t+1)} \frac{1}{1+\varphi(t)} = 0.$$

Since $\frac{1}{1+\varphi(t)} = 1 + \mathcal{O}(\varphi(t))$ as $t \to \infty$),

$$\varphi(t+1) = -\frac{1}{t(t+1)}(1 + \mathcal{O}(\varphi(t))), \quad (t \to \infty).$$

So

$$\varphi(t) = \mathcal{O}\left(\frac{1}{t^2}\right), \quad (t \to \infty).$$

Then

$$v(t+1) = \left(1 + \mathcal{O}\left(\frac{1}{t^2}\right)\right) v(t)$$

and Theorem 5.3.9 implies

$$v(t) \sim C, \quad (t \to \infty)$$

for some constant C. Finally, we have

$$u_1(t) \sim C(t-1)!, \quad (t \to \infty).$$

Next, set

$$\frac{v(t+1)}{v(t)} = \psi(t),$$

where $\psi(t) \to 0$ as $t \to \infty$. Then ψ satisfies

$$\psi(t) = \frac{1}{t(t+1)} + \psi(t)\psi(t+1)$$

or

$$\psi(t) = \frac{1}{t(t+1)}\left(1 + \mathcal{O}\left(\frac{1}{t^2}\right)\right), \quad (t \to \infty).$$

It follows that

$$v(t+1) = \frac{1}{t(t+1)}\left(1 + \mathcal{O}\left(\frac{1}{t^2}\right)\right) v(t), \quad (t \to \infty).$$

Iteration and Theorem 5.3.9 yield a constant D *so that*

$$v(t) \sim \frac{D}{t!(t-1)!}, \quad (t \to \infty)$$

and we obtain a second solution $u_2(t)$ *that satisfies*

$$u_2(t) \sim \frac{D}{t!}, \quad (t \to \infty).$$

Example 5.3.11. *Consider* $u(t+2) - 3tu(t+1) + 2t^2u(t) = 0$.

If we seek a rapidly increasing solution, it is not clear in this case that any term is asymptotically smaller than the others. In fact, a growth rate of $t!$ *would roughly balance the size of the three terms. Let*

$$u(t) = t!v(t).$$

Then $v(t)$ *satisfies*

$$v(t+2) - \left(3 - \frac{6}{t+2}\right) v(t+1) + 2\left(1 - \frac{3t+2}{(t+1)(t+2)}\right) v(t) = 0$$

which is of Poincaré type. By Perron's Theorem, there are independent solutions v_1, v_2 *so that*

$$\frac{v_1(t+1)}{v_1(t)} \to 1, \quad \frac{v_2(t+1)}{v_2(t)} \to 2$$

as $t \to \infty$. *Let* $\frac{v_1(t+1)}{v_1(t)} = 1 + \varphi(t)$ *so that* $\lim_{t\to\infty} \varphi(t) = 0$. *A short computation leads to*

$$\varphi(t+1) - 2\varphi(t) = -\frac{2}{(t+1)(t+2)} + \mathcal{O}\left(\frac{\varphi(t)}{t}\right) + \mathcal{O}\left(\varphi^2(t)\right), \quad (t \to \infty).$$

If we call the righthand side of the preceding equation $r(t)$, *then the general solution is*

$$\varphi(t) = 2^{t-1}\left(C + \sum_{s=1}^{t-1} \frac{r(s)}{2^s}\right).$$

To satisfy the condition $\lim_{t\to\infty}\varphi(t)=0$*, choose* $C=-\sum_{s=1}^{\infty}\frac{r(s)}{2^s}$*, then*

$$\varphi(t)=-\sum_{s=t}^{\infty}\frac{r(s)}{2^{s-t+1}}$$

and

$$|\varphi(t)|\leq\max_{s\geq t}|r(s)|\sum_{s=t}^{\infty}\frac{1}{2^{s-t+1}}$$

or

$$|\varphi(t)|\leq\max_{s\geq t}|r(s)|.$$

It follows that $\varphi(t)=\mathcal{O}\left(1/t^2\right)$ *as* $t\to\infty$*,* $v_1(t)\sim C_1$*,* $(t\to\infty)$*. A solution* $u_1(t)$ *of the original equation then satisfies*

$$u_1(t)\sim C_1t!,\quad (t\to\infty).$$

Now, set

$$\frac{v_2(t+1)}{v_2(t)}=2+\psi(t)$$

with $\psi(t)<<1(t\to\infty)$*. We find*

$$\psi(t+1)-\frac{\psi(t)}{2}=\frac{3t+4}{(t+1)(t+2)}+\mathcal{O}\left(\psi^2(t)\right),\quad (t\to\infty).$$

Since

$$\frac{3t+4}{(t+1)(t+2)}=\frac{3}{t}+\mathcal{O}\left(\frac{1}{t^2}\right),\quad (t\to\infty),$$

the general solution is

$$\psi(t)=2^{1-t}\left[C+\sum_{s=1}^{t-1}2^s\left(\frac{3}{s}+\mathcal{O}\left(\frac{1}{s^2}\right)+\mathcal{O}\left(\psi^2(s)\right)\right)\right],\quad (t\to\infty).$$

Also,

$$\sum_{s=1}^{t-1}2^s\left(\frac{3}{s}\right)=\frac{3}{t}2^t\left[1+\mathcal{O}\left(\frac{1}{t}\right)\right],\quad (t\to\infty).$$

Since

$$\sum_{s=1}^{t-1}2^s\left(\frac{1}{s^2}\right)=\mathcal{O}\left(\frac{2^t}{t^2}\right),\quad (t\to\infty),$$

we have

$$\psi(t)=\frac{6}{t}+\mathcal{O}\left(\frac{1}{t^2}\right),\quad (t\to\infty).$$

Then, v_2 satisfies

$$v_2(t+1) = 2\left(\frac{t+3}{t}\right)\left[1+\mathcal{O}\left(\frac{1}{t^2}\right)\right]v_2(t)$$

and we find by iteration

$$v_2(t) \sim C_2 t^3 2^{t-1}, \quad (t \to \infty).$$

So finally

$$u_2(t) \sim C_2 2^{t-1}(t+3)!, \quad (t \to \infty).$$

Let Us Sum Up:

In this section, we have discussed the following concepts:

1. Asymptotic behavior of solutions of homogeneous linear equations
2. Poincare type difference equation
3. Poincare's Theorem
4. Perron's Theorem

Check your Progress:

1. If $\frac{u(t+1)}{u(t)} \to \lambda \quad (t \to \infty)$ and if $\lambda \neq 0$, then ....

   (A) $u(t) = \pm\lambda^t e^{z(t)}$ with $z(t) << t, \quad (t \to \infty)$

   (B) $u(t) = \pm\lambda^t e^{z(t)}$ with $t << z(t), \quad (t \to \infty)$

   (C) $u(t) = \pm\lambda^t e^{z(t)}$ with $z(t) \sim t, \quad (t \to \infty)$

   (D) None of these

2. If $\frac{u(t+1)}{u(t)} \to \lambda \quad (t \to \infty)$ and if $\lambda = 0$, then ....

   (A) $|u(t)| = e^{-z(t)}$ with $z(t) << t, \quad (t \to \infty)$

   (B) $|u(t)| = e^{-z(t)}$ with $z(t) \sim t, \quad (t \to \infty)$

   (C) $|u(t)| = e^{-z(t)}$ with $z(t) >> t, \quad (t \to \infty)$

   (D) None of these

3. Which of the following functions doesn't satisfy $\lim_{t\to\infty}\frac{u(t+1)}{u(t)}=1$?

   (A) 5 (B) t (C) $\log t$ (D) None of these

Unit Summary:

Approximations of solutions to difference equations for large values of the independent variable are studied in this unit. The asymptotic behavior of solutions of homogeneous linear equations is discussed using Poincare's theorem and Perron's theorem.

Glossary:

- $y(t) \sim z(t), \quad (t \to \infty)$ - $y(t)$ is asymptotic to $z(t)$ as t tends to infinity
- $u(t) << v(t), \quad (t \to \infty)$ - $u(t)$ is much smaller than $v(t)$ as t tends to infinity
- $u(t) = \mathcal{O}(v(t)), \quad (t \to \infty)$ - $u(t)$ is big oh of $v(t)$ as t tends to infinity

Self-Assessment Questions:

1. Verify the following asymptotic relation: $\frac{1}{t^2+2t-7} \sim \frac{1}{t^2}, \quad (t \to \infty)$.

2. Verify

   (a) $5x^2 \sin 3x = \mathcal{O}\left(x^2\right), \quad (x \to \infty)$.

   (b) $\frac{1}{x-2} = \frac{1}{x}\left[1 + \mathcal{O}\left(\frac{1}{x}\right)\right], \quad (x \to \infty)$.

3. Give estimates of

   (a) $\int_1^\infty \frac{e^{-3t}}{t^{50}}dt$.

   (b) $\int_1^\infty \frac{e^{-50t}}{t^3}dt$.

4. Use Taylor's formula to obtain an asymptotic estimate for

$$\sum_{k=1}^{n-1} \frac{1}{(2k)!}, \quad (n \to \infty)$$

5. Investigate the asymptotic behavior of the solutions of

$$t^2u(t+2) - 3tu(t+1) + 2u(t) = 0$$

   as $t \to \infty$.

Exercises:

1. Verify $\tan(1/t^2) << \frac{10}{t}, \quad (t \to \infty)$.

2. Show that $\sqrt{t^2+1} = t\left[1 + \frac{1}{2t^2} + \mathcal{O}\left(\frac{1}{t^4}\right)\right], \quad (t \to \infty)$.

3. Use integration by parts to show
$$\int_0^\infty \frac{e^{-t}}{x+t}dt = \frac{1}{x}\left(1 - \frac{1}{x} + \mathcal{O}\left(\frac{1}{x^2}\right)\right), \quad (x \to \infty)$$

4. Verify
   (a) $\sum_{k=1}^n k! = n!\left[1 + \mathcal{O}\left(\frac{1}{n}\right)\right], (n \to \infty)$.
   (b) $\sum_{k=1}^n k! = n!\left[1 + \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right]$, $(n \to \infty)$.

5. Verify Wallis' formula:
$$\frac{\pi}{2} = \lim_{n\to\infty}\left[\frac{2\cdot 4\cdot\cdots\cdot 2n}{1\cdot 3\cdot\cdots\cdot(2n-1)}\right]^2\frac{1}{2n+1}$$
   (Hint: first show that
$$\int_0^{\frac{\pi}{2}} \sin^{2n-1} x dx = \frac{2\cdot 4\cdots(2n-2)}{1\cdot 3\cdots(2n-1)}$$
   and
$$\int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots(2n)}\frac{\pi}{2}$$
   Next, integrate the inequalities $\sin^{2n+1} x \leq \sin^{2n} x \leq \sin^{2n-1} x$, which hold on the interval $\left[0, \frac{\pi}{2}\right]$.)

6. Show that if $\lim_{t\to\infty}\frac{u(t+1)}{u(t)} = \lambda > 0$, then for each δ in $(0,\lambda)$, $(\lambda-\delta)^t << |u(t)| << (\lambda+\delta)^t, (t \to \infty)$.

Answers for Check your Progress:

Section 5.1	1. (A)	2. (B)	3. (C)
Section 5.2	1. (C)	2. (D)	3. (C)
Section 5.3	1. (A)	2. (C)	3. (D)

References:

1. W.G. Kelley and A.C. Peterson, "Difference Equations", 2nd Edition, Academic Press, New York, 2001.

Suggested Readings:

1. R.P. Agarwal, “Difference Equations and Inequalities”, 2nd Edition, Marcel Dekker, New York, 2000.

2. S.N. Elaydi, “An Introduction to Difference Equations”, 3rd Edition, Springer, India, 2008.

3. R. E. Mickens, “Difference Equations”, 3rd Edition, CRC Press, 2015.